Lecture 3: Portfolio Management-A Risky and a Riskless Asset.

I. Reading.
II. Investor Preferences.
III. Expected Portfolio Return: General Formula
IV. Standard Deviation of Portfolio Return: One Risky Asset and a Riskless Asset.
V. Graphical Depiction: Portfolio Expected Return and Standard Deviation.
VI. Portfolio Management: One Risky Asset and a Riskless Asset.
Lecture 3: Portfolio Management—A Risky and a Riskless Asset.

I. Reading.
A. BKM, Chapter 6: read this chapter (though Section 6.1 is more detailed than is needed); ignore the Appendices.
B. BKM, Chapter 7: skim Sections 7.1 and 7.2; read Section 7.3; read lightly Sections 7.4 and 7.5.

II. Investor Preferences.
A. Summarizing Tastes and Preferences.
   1. For the moment, assume a one period setting.
   2. For certain return distributions (e.g., multivariate normal), individual preferences can be completely characterized by:
      a. Expected Return over the Period, $E[R]$.
      b. Standard Deviation of Return over the Period, $\sigma[R]$.
   3. In other words, individuals only care about their expected portfolio return and about their portfolio’s standard deviation.
B. Risk Aversion.
   1. One of the cornerstones of modern finance is that individuals are risk averse (and prefer more to less).
   2. For any risk averse individual, the following is true:
      a. For a given expected portfolio return, prefer a portfolio with a lower standard deviation of return.
      b. For a given standard deviation of portfolio return, prefer a portfolio with a higher expected return.
Class Illustration:

<table>
<thead>
<tr>
<th>Which of each pair do you prefer?</th>
<th>Investment</th>
<th>Payoff Good State</th>
<th>Payoff Bad State</th>
<th>E[R]</th>
<th>σ[R]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2M</td>
<td>1M</td>
<td>50%</td>
<td>50%</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>1.75M</td>
<td>0.75M</td>
<td>25%</td>
<td>50%</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>2M</td>
<td>1M</td>
<td>50%</td>
<td>50%</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>2.4M</td>
<td>0.6M</td>
<td>50%</td>
<td>90%</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>2M</td>
<td>1M</td>
<td>50%</td>
<td>50%</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>2.4M</td>
<td>0.8M</td>
<td>60%</td>
<td>80%</td>
<td></td>
</tr>
</tbody>
</table>
3. Use indifference curves to represent an individual’s tastes and preferences:

a. At all points on an indifference curve, the investor enjoys the same level of utility.

b. In \{Standard Deviation of Return, Expected Return\} space, a risk averse individual’s indifference curves have positive slopes: Since a risk averse individual likes mean but dislikes standard deviation, the only way the individual can accept more standard deviation and maintain the same level of utility is if she is given a higher expected return.

c. For any individual, as you move north in \{\sigma[R], E[R]\} space, utility is increasing.

d. For any individual, her indifference curves can not cross since that would imply that a particular \{\sigma[R], E[R]\} combination was associated with two levels of utility.

4. However, the trade-off between risk and return for any two risk averse individuals may be completely different (see individuals Y and Z above).

a. Individual Y is more risk averse than Z since at any point in \{\sigma[R], E[R]\} space, Y’s indifference curve has a steeper slope.
III. Expected Portfolio Return: General Formula

A. Formula: holds for any number of assets and with or without the risky asset as one of the assets:

\[ E[R_p(t)] = \omega_{1,p} E[R_1(t)] + \omega_{2,p} E[R_2(t)] + \ldots + \omega_{N,p} E[R_N(t)] \]

where

- \( N \) is the number of assets in the portfolio;
- \( E[R_i(t)] \) is the expected return on asset \( i \) in period \( t \);
- \( \omega_{ip} \) is the weight of asset \( i \) in the portfolio \( p \) at the start of period \( t \);
- \( E[R_p(t)] \) is the expected return on portfolio \( p \) in period \( t \).

B. Example 1 (cont): Consider a portfolio with 80% invested in Ford and the remaining 20% invested in T-bills.

\[ E[R_p] = 0.8 \times 9.6\% + 0.2 \times 5\% = 8.68\%. \]

C. Example 2 (cont): Consider a portfolio formed at the end of January 1997 with 60% invested in the small firm portfolio and the remaining 40% invested in 1 month T-bills.

1. What is the portfolio’s expected return ignoring DP(start Feb)?

\[ E[R_p] = 0.6 \times 1.912\% + 0.4 \times 0.323\% = 1.2764\%. \]

2. What is the portfolio’s expected return using DP(start Feb)?

\[ E[R_p] = 0.6 \times \left(-1.509\%\right) + 0.4 \times 0.323\% = -0.7762\%. \]

3. Using the starting DP to help determine expected return can make a big difference.
IV. Standard Deviation of Portfolio Return: One Risky Asset and a Riskless Asset.

A. Formula: holds when one asset is risky and the other is riskless:

\[ \sigma[R_p(t)] = |\omega_{i,p}| \sigma[R_i(t)] \]

where

- \( \sigma[R_i(t)] \) is the standard deviation of return on risky asset \( i \) in period \( t \);
- \( |\omega_{i,p}| \) is the absolute value of the weight of asset \( i \) in the portfolio \( p \);
- \( \sigma[R_p(t)] \) is the standard deviation of return on portfolio \( p \) in period \( t \).

B. Example 1 (cont): Consider the portfolio with 80% invested in Ford and the remaining 20% invested in T-bills.

\[
\sigma[R_p] = |\omega_{Ford,p}| \sigma[R_{Ford}]
\]

\[
= 0.8 \times 15.5897\% = 12.4718\%.
\]

C. Example 2 (cont): Consider the portfolio formed at the end of January 1997 with 60% invested in the small firm portfolio and the remaining 40% invested in 1 month T-bills.

1. What is the portfolio’s standard deviation ignoring DP (start Feb)?

\[
\sigma[R_p] = |\omega_{Small,p}| \sigma[R_{Small}]
\]

\[
= 0.6 \times 3.711\% = 2.2266\%.
\]

2. What is the portfolio’s standard deviation using DP (start Feb)?

\[
\sigma[R_p] = |\omega_{Small,p}| \sigma[R_{Small}]
\]

\[
= 0.6 \times 3.549\% = 2.1294\%.
\]

3. Portfolio standard deviation is largely unaffected by using the starting DP to predict return.

4. Note that using these formulas are just as easy for the real data as for the mock data.
V. Graphical Depiction: Portfolio Expected Return and Standard Deviation.

A. Example 2 (cont): The standard deviation of return on a portfolio consisting of the small firm asset and T-bills and its expected return can be indexed by the weight of the small firm asset in the portfolio:

1. Ignoring DP(start Feb):

<table>
<thead>
<tr>
<th>( \omega_{\text{Small},p} )</th>
<th>( \omega_{\text{T-bill},p} )</th>
<th>( \sigma[R_p(t)] )</th>
<th>( E[R_p(t)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>1.2</td>
<td>0.742%</td>
<td>0.005%</td>
</tr>
<tr>
<td>0.0</td>
<td>1.0</td>
<td>0.000%</td>
<td>0.323%</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>0.742%</td>
<td>0.641%</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>1.484%</td>
<td>0.959%</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>2.227%</td>
<td>1.277%</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>2.969%</td>
<td>1.595%</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>3.711%</td>
<td>1.912%</td>
</tr>
<tr>
<td>1.2</td>
<td>-0.2</td>
<td>4.453%</td>
<td>2.230%</td>
</tr>
</tbody>
</table>
2. Using DP (start Feb):

<table>
<thead>
<tr>
<th>$\omega_{\text{Small}, p}$</th>
<th>$\omega_{\text{T-bill}, p}$</th>
<th>$\sigma[R_p(t)]$</th>
<th>$E[R_p(t)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>1.2</td>
<td>0.710</td>
<td>0.689</td>
</tr>
<tr>
<td>0.0</td>
<td>1.0</td>
<td>0.000</td>
<td>0.323</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>0.710</td>
<td>-0.043</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>1.420</td>
<td>-0.410</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>2.129</td>
<td>-0.776</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>2.839</td>
<td>-1.143</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>3.549</td>
<td>-1.509</td>
</tr>
<tr>
<td>1.2</td>
<td>-0.2</td>
<td>4.259</td>
<td>-1.875</td>
</tr>
</tbody>
</table>

Portfolio of the Small Firm Asset and T-bills: Using DP

$\omega_{\text{Small}, p}$ decreases as $E[R_p]$ increases

{$\sigma[R_{\text{Small}}]$, $E[R_{\text{Small}}]$} marked by +
VI. Portfolio Management: One Risky Asset and a Riskless Asset.
A. How would a risk averse investor choose the portfolio weights for a portfolio consisting solely of the riskless asset and a given risky asset?
   1. For any point on the negative sloped part of the curve, a risk averse individual is going to prefer at least one point on the positive sloped part of the curve (the one with the same standard deviation and a higher expected return).
   2. So if the expected return on a risky asset exceeds the riskless rate, an individual forming a portfolio using only that asset and the riskless asset will not want to short sell the risky asset (not want $\omega_{\text{Risky,p}} < 0$); but the individual may want to buy it on margin (may want $\omega_{\text{Risky,p}} > 1$).
      a. Example 2 (cont): Combining the small firm portfolio with T-bills ignoring DP.
   3. So if the expected return on a risky asset is less than the riskless rate, an individual forming a portfolio using only that asset and the riskless asset will want to short sell the risky asset (will want $\omega_{\text{Risky,p}} < 0$).
      a. Example 2 (cont): Combining the small firm portfolio with T-bills using DP.
4. The exact weight that the individual wants to hold of the risky asset depends on her attitudes to risk; different individuals will choose to hold different amounts of the risky asset.
   a. Example 2 (cont): Combining the small firm asset with T-bills ignoring DP.
      (1) Y wants to hold positive amounts of both the small firm asset and T-bills: $0 < \omega_{\text{Small},p} < 1$.
      (2) Z wants to buy the small firm asset on margin: $\omega_{\text{Small},p} > 1$

5. The positive sloped line is called the Capital Allocation Line (CAL).
B. If a risk averse investor could use either risky asset A or risky asset B in combination with the riskless asset, how would the investor decide whether to use risky asset A or to use risky asset B?

1. Example 2 (cont): If a risk averse investor could use either the small firm asset or Microsoft in combination with the riskless asset to form a portfolio, how would the investor decide whether to use the small firm asset or to use Microsoft (ignoring DP)?

2. Irrespective of risk preferences, the individual prefers the risky asset whose CAL has the highest slope.
   a. For any point on the lower sloped line, a risk averse investor prefers at least one point on the higher sloped line (the point with the same standard deviation but a higher expected return).
   b. Example 2 (cont):
3. In general, the slope of any risky asset $i$’s CAL is given by
\[ \text{slope}[\text{CAL}_i] = \frac{|E[R_i] - R_f|}{\sigma[R_i]} . \]

4. Example 2 (cont): Calculate the slope of the CAL for the small firm asset and Microsoft ignoring DP:
   a. \( \text{slope}[\text{CAL}_{\text{Small}}] = \frac{1.912-0.323}{3.711} = 0.428; \)
   \( \text{slope}[\text{CAL}_{\text{Msft}}] = \frac{3.126-0.323}{8.203} = 0.342; \)
   \( \text{slope}[\text{CAL}_{\text{Small}}] > \text{slope}[\text{CAL}_{\text{Msft}}] . \)
   b. A risk averse individual’s preference for using the risky asset with the highest-sloped CAL (the small firm asset) can also be seen by examining the behavior of $Y$ and $Z$:
   
   c. Both individual $Y$ and individual $Z$ can attain higher utility holding the small firm asset rather than Microsoft in combination with the riskless asset.
Lecture 3: Portfolio Management-2 Risky Assets and a Riskless Asset.

I. Reading.
II. Standard Deviation of Portfolio Return: Two Risky Assets.
III. Graphical Depiction: Two Risky Assets.
IV. Impact of Correlation: Two Risky Asset Case.
V. Portfolio Choice: the Two Risky Asset Portfolio.
VI. Portfolio Choice: Combining the Two Risky Asset Portfolio with the Riskless Asset.
VII. Applications.
Lecture 3: Portfolio Management-2 Risky Assets and a Riskless Asset.

I. Reading.
   A. BKM, Chapter 8: read Sections 8.1 to 8.3.

II. Standard Deviation of Portfolio Return: Two Risky Assets.
   A. Formula:

\[
\sigma^2[R_p(t)] = \omega_{1,p}^2 \sigma[R_1(t)]^2 + \omega_{2,p}^2 \sigma[R_2(t)]^2 + 2 \omega_{1,p} \omega_{2,p} \sigma[R_1(t), R_2(t)]
\]

where

- \( \sigma[R_1(t), R_2(t)] \) is the covariance of asset 1’s return and asset 2’s return in period t;
- \( \omega_{i,p} \) is the weight of asset i in the portfolio p;
- \( \sigma^2[R_p(t)] \) is the variance of return on portfolio p in period t.

   B. Example 2 (cont): Consider a portfolio formed at the start of February 1997 with 60% invested in the small firm asset and 40% in Microsoft.
   1. Use data in Lecture 2 pp.29-32.
   2. What is the portfolio’s standard deviation ignoring DP?

\[
\sigma^2[R_p] = 0.6^2 \times 3.711^2 + 0.4^2 \times 8.203^2 + 2 \times 0.6 \times 0.4 \times 12.030
\]

\[
= 0.36 \times 13.75 + 0.16 \times 67.22 + 5.774 = 21.498.
\]

\[
\sigma[R_p] = \sqrt{\sigma^2[R_p]} = \sqrt{21.498} = 4.637.
\]

3. Obtain expected portfolio return using the formula on page 4 of Lecture 3.

\[
E[R_p] = \omega_{Small,p} E[R_{Small}] + \omega_{Msft,p} E[R_{Msft}]
\]

\[
= 0.6 \times 1.912 + 0.4 \times 3.126
\]

\[
= 2.398
\]
III. Graphical Depiction: Two Risky Assets.

A. The standard deviation of return on a portfolio consisting of the small firm asset and Microsoft and its expected return can be indexed by the weight of the small firm asset in the portfolio. The curve is known as the portfolio possibility curve.

<table>
<thead>
<tr>
<th>( \omega_{\text{Small},p} )</th>
<th>( \omega_{\text{Msft},p} )</th>
<th>( \sigma[R_p(t)] )</th>
<th>( E[R_p(t)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>1.2</td>
<td>9.574%</td>
<td>3.369%</td>
</tr>
<tr>
<td>0.0</td>
<td>1.0</td>
<td>8.203%</td>
<td>3.126%</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>6.889%</td>
<td>2.883%</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>5.675%</td>
<td>2.641%</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>4.637%</td>
<td>2.398%</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>3.919%</td>
<td>2.155%</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>3.711%</td>
<td>1.912%</td>
</tr>
<tr>
<td>1.2</td>
<td>-0.2</td>
<td>4.093%</td>
<td>1.670%</td>
</tr>
</tbody>
</table>

Portfolio of the Small Firm Asset and Msft: Ignoring DP \( \{\sigma[R_{\text{Small}}], E[R_{\text{Small}}]\} \) marked by + and \( \{\sigma[R_{\text{Msft}}], E[R_{\text{Msft}}]\} \) marked by x.
IV. Impact of Correlation: Two Risky Asset Case.

A. Standard Deviation Formula.

1. Can be rewritten in terms of correlation rather than covariance (using the definition of correlation):

\[
\sigma^2[R_p(t)] = \omega^2_{1,p} \sigma[R_1(t)]^2 + \omega^2_{2,p} \sigma[R_2(t)]^2 + 2 \omega_{1,p} \omega_{2,p} \rho[R_1(t), R_2(t)] \sigma[R_1(t)] \sigma[R_2(t)]
\]

where \( \rho[R_1(t), R_2(t)] \) is the correlation of asset 1’s return and asset 2’s return in period t;

2. For a given portfolio with \( \omega_{1,p}, \omega_{2,p} > 0 \) and fixed \( \sigma[R_1(t)] \) and \( \sigma[R_2(t)] \), decreases as \( \rho[R_1(t), R_2(t)] \) decreases.

B. Example 2 (cont):

a. Suppose the \( \mathbb{E}[R] \) and \( \sigma[R] \) for the small firm asset and for Microsoft remain the same but the correlation between the two assets is allowed to vary:

---

Portfolio of the Small Firm Asset and Msft: Ignoring DP

\( \rho = \rho[R_{Small}, R_{Msft}] \)

\( \{\sigma[R_{Small}], \mathbb{E}[R_{Small}]\} \) marked by +, \( \{\sigma[R_{Msft}], \mathbb{E}[R_{Msft}]\} \) marked by x

---

\( \rho = 0.395 \)

\( \rho = 1 \)

\( \rho = 0 \)

\( \rho = -1 \)
V. Portfolio Choice: the Two Risky Asset Portfolio.

A. A risk averse investor is not going to hold any combination of the two risky assets on the negative sloped portion of the portfolio possibility curve.
   1. So the negative-sloped portion is known as the inefficient region of the curve.
   2. And the positive-sloped portion is known as the efficient region of the curve.

B. The exact position on the efficient region that an individual holds depends on her tastes and preferences.

C. Example 2 (cont): The portfolio possibility curve for the small firm asset and Microsoft can be divided into its efficient and inefficient regions.
   1. Any risk averse individual combining the small firm asset with Microsoft wants to lie in the efficient region: so wants to invest a positive fraction of her portfolio in Microsoft.

---

**Portfolio of the Small Firm Asset and Msft: Ignoring DP**

Efficient vs Inefficient

\( \{\sigma[R_{Small}], E[R_{Small}]\} \) marked by +, \( \{\sigma[R_{Msft}], E[R_{Msft}]\} \) marked by x

---

1. 0 2 4 6 8 10

- 1 2 3 4

\( \sigma[R] \)

\( E[R] \)
VI. Portfolio Choice: Combining the Two Risky Asset Portfolio with the Riskless Asset.

A. Two-step Decision Process:

1. What is the preferred weights of the two risky assets in the risky portfolio?
   a. all risk averse individuals want access to the CAL with the largest slope; this involves combining the riskless asset with the same risky portfolio (\( \bullet \) in the graph below).
   b. this same risky portfolio is the one whose CAL is tangential to the portfolio opportunity curve; this is why \( \bullet \) is known as the tangency portfolio (denoted T).
   c. Can calculate the weight of risky asset 1 in the tangency portfolio T using the following formula:

\[
\omega_{1,T} = \frac{\sigma[R_2]^2 \ E[r_1] - \sigma[R_1, R_2] \ E[r_2]}{\left\{ \sigma[R_2]^2 \ E[r_1] - \sigma[R_1, R_2] \ E[r_2] \right\} + \left\{ \sigma[R_1]^2 \ E[r_2] - \sigma[R_1, R_2] \ E[r_1] \right\}}
\]

where \( r_i = R_i - R_f \) is the excess return on asset i (in excess of the riskless rate).

2. What is the preferred weights of the risky portfolio T and the riskless asset in the individual’s portfolio?
   a. the weight of T (\( \bullet \)) in an individual’s portfolio \( \omega_{r,p} \) depends on the individual’s tastes and preferences.
B. Example 2 (cont): Suppose I form a portfolio (ignoring DP) of the small firm asset, Microsoft and T-bills.

1. What is the preferred weights of the two risky assets in the risky portfolio?
   a. Graph suggests that the risky asset portfolio I want to hold has positive weights invested in the small firm asset and in Microsoft (since \( \bullet \) is between + and x on the portfolio possibility curve for the small firm asset and Microsoft).
   b. Can calculate the weight of the small firm asset in the tangency portfolio using the following formula:

   \[
   \omega_{\text{Small}, T} = \frac{\sigma[R_{\text{Msft}}]^2 E[r_{\text{Small}}] - \sigma[R_{\text{Small}}, R_{\text{Msft}}] E[r_{\text{Msft}}]}{\sigma[R_{\text{Small}}]^2 E[r_{\text{Msft}}] - \sigma[R_{\text{Small}}, R_{\text{Msft}}] E[r_{\text{Msft}}] + \{\sigma[R_{\text{Msft}}]^2 E[r_{\text{Small}}] - \sigma[R_{\text{Small}}, R_{\text{Msft}}] E[r_{\text{Msft}}]\}}
   \]

   c. Now (using Lecture 2 pp.29-32)

   \[
   \begin{align*}
   E[r_{\text{Small}}] &= 1.912 - 0.323 = 1.589. \\
   E[r_{\text{Msft}}] &= 3.126 - 0.323 = 2.803. \\
   \sigma[R_{\text{Small}}]^2 &= 3.711 \times 3.711 = 13.772. \\
   \sigma[R_{\text{Msft}}]^2 &= 8.203 \times 8.203 = 67.289. \\
   \sigma[R_{\text{Msft}}, R_{\text{Small}}] &= 12.030.
   \end{align*}
   \]

   \[
   \omega_{\text{Small}, T} = \frac{67.289 \times 1.589 - 12.030 \times 2.803}{67.289 \times 1.589 - 12.030 \times 2.803 + \{13.772 \times 2.803 - 12.030 \times 1.589\}}
   \]

   \[
   = \frac{73.202}{73.202 + 19.487} = 0.790.
   \]

   2. What is the preferred weights of the risky portfolio T and the riskless asset in the individual’s portfolio?
   a. Depends on the tastes and preferences of the particular individual.
3. Suppose Individual Y wants to invest 75% in the tangency portfolio (⊕) and 25% in T-bills. What is the weight of the small firm asset and of Microsoft in Y’s total portfolio?
   a. Use the following formula:

   \[ \omega_{i,p} = \omega_{i,T} \omega_{T,p} \]

   where
   \[ \omega_{i,T} \] is the weight of risky asset \( i \) in the tangency portfolio \( T \).
   \[ \omega_{i,p} \] is the weight of risky asset \( i \) in the total portfolio \( p \).
   \[ \omega_{T,p} \] is the weight of portfolio \( T \) in the total portfolio \( p \).

   b. So, the answer is:

   \[ \omega_{\text{Small},p} = \omega_{\text{Small},T} \omega_{T,p} = 0.79 \times 0.75 = 0.5925. \]

   \[ \omega_{\text{Msft},p} = \omega_{\text{Msft},T} \omega_{T,p} = 0.21 \times 0.75 = 0.1575. \]
VII. Applications.
   A. Adding a New Stock: The two-risky-asset formulas can be used to assess the impact of adding a new stock to a portfolio or varying the weight of an existing stock in the portfolio.
      1. Example 2 (cont): Above considered the impact of adding Microsoft to the small firm fund (ignoring DP).
   B. Asset Allocation between Two Broad Classes of Assets: The two-risky-asset formulas can be used to determine how much to invest in each of two broad asset classes.
      1. Example 2 (cont): If I intend to form a risky portfolio from the small firm asset and the S&P 500 (ignoring DP) and then combine that risky portfolio with the riskless asset, what weight will the small firm asset have in the risky portfolio?
2. Can see from the graph that using historical data from 1/91 to 12/95 to approximate the return distribution for 2/97, would lead a risk averse individual to hold a risky portfolio with positive weights in both the small firm asset and in the S&P 500 (ω_{Small,T} = 0.6687 using formula above).

3. Weight of the small firm asset in the tangency portfolio ω_{Small,T} is sensitive to the E[R_{Small}]:
   a. As E[R_{Small}] declines so does ω_{Small,T} holding E[R_{S&P}], the standard deviations and correlation fixed:

<table>
<thead>
<tr>
<th>E[R_{Small}]</th>
<th>E[R_{S&amp;P}]</th>
<th>ω_{Small,T}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.912%</td>
<td>1.326%</td>
<td>66.87%</td>
</tr>
<tr>
<td>1.619%</td>
<td>1.326%</td>
<td>44.24%</td>
</tr>
<tr>
<td>1.326%</td>
<td>1.326%</td>
<td>19.09%</td>
</tr>
</tbody>
</table>

   b. This sensitivity of ω_{Small,T} to changes in E[R_{Small}] explains why the small firm effect is of interest to practitioners.

C. International Diversification: The two-risky-asset formulas can also be used when deciding how much to invest in an international equity fund and how much in a U.S. based fund.
Lecture 3: Portfolio Management - N Risky Assets and a Riskless Asset

I. Reading.
II. Standard Deviation of Portfolio Return: N Risky Assets.
III. Effect of Diversification.
IV. Opportunity Set: N Risky Assets.
V. Portfolio Choice: N Risky Assets and a Riskless Asset
Lecture 3: Portfolio Management - N Risky Assets and a Riskless Asset

I. Reading.
A. BKM, Chapter 8, Sections 8.4 and 8.5 and Appendix 8.A.

II. Standard Deviation of Portfolio Return: N Risky Assets.
1. Formula.

\[ \sigma^2[R_p(t)] = \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_{i,p} \omega_{j,p} \sigma[R_i(t), R_j(t)] \]

where
\[ \sigma[R_i(t), R_j(t)] \] is the covariance of asset i’s return and asset j’s return in period t;
\[ \omega_{i,p} \] is the weight of asset i in the portfolio p;
\[ \sigma^2[R_p(t)] \] is the variance of return on portfolio p in period t.

2. The formula says that \[ \sigma^2[R_p(t)] \] is equal to the sum of the elements in the following N x N matrix.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>j ...</th>
<th>N-1</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\omega_{1,p} \omega_{1,p} / \sigma[R_1, R_1])</td>
<td>(\omega_{1,p} \omega_{2,p} / \sigma[R_1, R_2])</td>
<td>(\omega_{1,p} \omega_{3,p} / \sigma[R_1, R_3])</td>
<td>(\omega_{1,p} \omega_{4,p} / \sigma[R_1, R_4, R_N])</td>
<td>(\omega_{1,p} \omega_{N,p} / \sigma[R_1, R_N])</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(\omega_{2,p} \omega_{1,p} / \sigma[R_2, R_1])</td>
<td>(\omega_{2,p} \omega_{2,p} / \sigma[R_2, R_2])</td>
<td>(\omega_{2,p} \omega_{3,p} / \sigma[R_2, R_3])</td>
<td>(\omega_{2,p} \omega_{4,p} / \sigma[R_2, R_4, R_N])</td>
<td>(\omega_{2,p} \omega_{N,p} / \sigma[R_2, R_N])</td>
<td></td>
</tr>
<tr>
<td>...i...</td>
<td>(\omega_{i,p} \omega_{1,p} / \sigma[R_i, R_1])</td>
<td>(\omega_{i,p} \omega_{2,p} / \sigma[R_i, R_2])</td>
<td>(\omega_{i,p} \omega_{3,p} / \sigma[R_i, R_3])</td>
<td>(\omega_{i,p} \omega_{4,p} / \sigma[R_i, R_4, R_N])</td>
<td>(\omega_{i,p} \omega_{N,p} / \sigma[R_i, R_N])</td>
<td></td>
</tr>
<tr>
<td>N-1</td>
<td>(\omega_{N-1,p} \omega_{1,p} / \sigma[R_{N-1}, R_1])</td>
<td>(\omega_{N-1,p} \omega_{2,p} / \sigma[R_{N-1}, R_2])</td>
<td>(\omega_{N-1,p} \omega_{3,p} / \sigma[R_{N-1}, R_3])</td>
<td>(\omega_{N-1,p} \omega_{4,p} / \sigma[R_{N-1}, R_4, R_N])</td>
<td>(\omega_{N-1,p} \omega_{N,p} / \sigma[R_{N-1}, R_N])</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>(\omega_{N,p} \omega_{1,p} / \sigma[R_N, R_1])</td>
<td>(\omega_{N,p} \omega_{2,p} / \sigma[R_N, R_2])</td>
<td>(\omega_{N,p} \omega_{3,p} / \sigma[R_N, R_3])</td>
<td>(\omega_{N,p} \omega_{4,p} / \sigma[R_N, R_4, R_N])</td>
<td>(\omega_{N,p} \omega_{N,p} / \sigma[R_N, R_N])</td>
<td></td>
</tr>
</tbody>
</table>

a. Notice that there are \(N^2\) terms.
b. The diagonal elements are the variance terms since \(\sigma^2[R_i(t)] = \sigma[R_i(t), R_i(t)];\) so there are \(N\) variance terms and \((N-1)N\) covariance terms.
c. Notice that this formula specializes to the formula used above for the two asset case:

$$\sigma^2[R_p(t)] = \omega_{1,p}^2 \sigma[R_1(t)]^2 + \omega_{2,p}^2 \sigma[R_2(t)]^2 + 2 \omega_{1,p} \omega_{2,p} \sigma[R_1(t), R_2(t)]$$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\omega_{1,p}$ $\omega_{1,p}$ $\sigma[R_1, R_1]$</td>
<td>$\omega_{1,p} \omega_{2,p}$ $\sigma[R_1, R_2]$</td>
</tr>
<tr>
<td>2</td>
<td>$\omega_{2,p}$ $\omega_{1,p}$ $\sigma[R_2, R_1]$</td>
<td>$\omega_{2,p} \omega_{2,p}$ $\sigma[R_2, R_2]$</td>
</tr>
</tbody>
</table>
III. Effect of Diversification.
   A. Consider an equal weighted portfolio (So $\omega_{i,p} = 1/N$ for all $i$). For example, when $N=2$, an equal weighted portfolio has 50% in each asset.
   B. Suppose all assets have the same $E[R] = \bar{R}$ and $\sigma[R] = \sigma$ and have returns which are uncorrelated. Then, for the equal weighted portfolio:
      1. $N=2$:
         
         \[
         E[R_p(t)] = \frac{1}{2} E[R_1(t)] + \frac{1}{2} E[R_2(t)] = \bar{R}.
         \]

         \[
         \sigma[R_p(t)]^2 = \left(\frac{1}{2}\right)^2 \sigma[R_1(t)]^2 + \left(\frac{1}{2}\right)^2 \sigma[R_2(t)]^2 = \frac{1}{2} \sigma^2.
         \]
      2. $N=3$:
         
         \[
         E[R_p(t)] = a E[R_1(t)] + a E[R_2(t)] + a E[R_3(t)] = \bar{R}.
         \]

         \[
         \sigma[R_p(t)]^2 = (a)^2 \sigma[R_1(t)]^2 + (a)^2 \sigma[R_2(t)]^2 + (a)^2 \sigma[R_3(t)]^2 = a^2 \sigma^2.
         \]
      3. Arbitrary $N$:
         
         \[
         E[R_p(t)] = \bar{R}.
         \]

         \[
         \sigma[R_p(t)]^2 = \sigma^2 / N.
         \]
      4. As $N$ increases:
         a. the variance of the portfolio declines to zero.
         b. the portfolio’s expected return is unaffected.
      5. This is known as the effect of diversification.
C. Suppose all assets have the same $\sigma[R] = \sigma$ and have returns which are correlated.

1. Formulas for expected return and standard deviation of return for the equal weighted portfolio can be written:

$$E[R_p(t)] = \text{average expected return}$$

$$\sigma^2[R_p(t)] = \sigma^2 \left[ \frac{1}{N} \right. 1 + \left. (1 - \frac{1}{N}) \text{average correlation} \right]$$

where

average expected return $= \frac{1}{N} \sum_{i=1}^{N} E[R_i(t)]$

average correlation $= \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \rho[R_i(t), R_j(t)]$.

2. As $N$ increases:
   a. Expected portfolio return is unaffected.
   b. Variance of portfolio return:
      (1) Expressed as a fraction of firm variance, portfolio variance converges to the average pairwise correlation between assets.

3. Shows the benefit of diversification depends on the correlation between the assets.

4. Can see that assets with low correlation maximize the diversification benefits.
D. Suppose assets have non-zero covariances and differing expected returns and standard deviations.

1. Formulas for expected portfolio return and standard deviation can be written:

\[ E[R_p(t)] = \text{average expected return} \]

\[ \sigma^2[R_p(t)] = \frac{1}{N} \text{average variance} + \left(1 - \frac{1}{N}\right) \text{average covariance} \]

where

\[ \text{average expected return} = \frac{1}{N} \sum_{i=1}^{N} E[R_i(t)] \]

\[ \text{average variance} = \frac{1}{N} \sum_{i=1}^{N} \sigma[R_i(t)]^2 \]

\[ \text{average covariance} = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \sigma[R_i(t), R_j(t)]. \]

2. As N increases:
   a. Expected portfolio return is unaffected.
   b. Variance of portfolio return:
      (1) First term (the unique/ firm specific/ diversifiable/ unsystematic risk) goes to zero.
      (2) Second term (the market/ systematic/ undiversifiable risk) remains.
         (a) When the assets are uncorrelated (the case above in III.B), this second term is zero.
IV. Opportunity Set: N Risky Assets.
   A. Set of Possible Portfolios.
      1. No longer a curve as in the two asset case.
      2. Instead, a set of curves.
   B. Minimum Variance Frontier.
      1. Since individuals are risk averse, can restrict attention to the set of portfolios with the lowest variance for a given expected return.
      2. This curve is known as the minimum variance frontier (MVF) for the risky assets.
      3. Every other possible portfolio is dominated by a portfolio on the MVF (lower variance of return for the same expected return).
      4. Example 2 (cont): Ignoring DP. The basic shape of the MVF is the same as the MVF for 4 individual stocks in this example (IBM, Apple, Microsoft and Nike) which is graphed below.
      5. Further, risk averse individuals would never hold a portfolio on the negative sloped portion of the MVF; so can restrict attention to the positive sloped portion. This portion is known as the efficient frontier.

---

**Minimum Variance Frontier MVF for IBM, Apple, Msft and Nike:**

Ignoring DP

**Efficient and Inefficient Frontiers, {\sigma[R_i], E[R_i]}s marked by x**

---

Efficient

Inefficient
C. Adding risky assets.
1. Adding risky assets to the opportunity set always causes the minimum variance frontier to shift to the left in \( \{\sigma[R], E[R]\} \) space. Why?
   a. For any given \( E[R] \), the portfolio on the MVF for the subset of risky assets is still feasible using the larger set of risky assets.
   b. Further, there may be another portfolio which can be formed from the larger set and which has the same \( E[R] \) but an even lower \( \sigma[R] \).

2. Example 2 (cont): Ignoring DP. MVF for IBM, Apple, Microsoft, Nike and ADM is to the left of the MVF for IBM, Apple, Microsoft and Nike excluding ADM. This happens even though ADM has an \( \{\sigma[R], E[R]\} \) denoted by \( \times \) which lies to the right of the MVF for the 4 stocks excluding ADM.
3. Example 2 (cont): Ignoring DP. MVF for all 8 assets (including the 3 funds or portfolios: small firm fund, S&P 500 fund and long-term government bonds fund) is to the left of the MVF for the 5 individual stocks (IBM, Apple, Microsoft, Nike and ADM).
V. Portfolio Choice: N Risky Assets and a Riskless Asset
   A. The analysis for the two risky asset and a riskless asset case applies here.
      1. Any risk averse individual combines the riskless asset with the risky portfolio whose Capital Accumulation Line has the highest slope.
      2. That risky portfolio is on the efficient frontier for the N risky assets and is known as the tangency portfolio ($\mathbf{\Theta}$): calculating the weights of assets in the tangency portfolio can be performed via computer.
      3. All risk averse individuals want to hold this tangency portfolio in combination with the riskless asset. The associated Capital Accumulation Line is the efficient frontier for the N risky assets and the riskless asset.
      4. Only the weights of the tangency portfolio and the riskless asset in an individual’s portfolio depend on the individual’s tastes and preferences.
      5. Example 2 (cont): Ignoring DP. If individuals can form a risky portfolio from the 5 individual assets and combine that risky portfolio with T-bills, then all individuals will hold $\mathbf{\Theta}$ as their risky portfolio. The weights of $\mathbf{\Theta}$ and T-bills in an individual’s portfolio will depend on that individual’s tastes and preferences.

Efficient Frontier for the 5 Stocks:
with and without T-bills: Ignoring DP
($\sigma[R_i]$, $E[R_i]$)s marked by $\times$ (stocks) and $+$
B. Adding risky assets to the set of available risky assets:
1. shifts the MVF for the risky assets to the left.
2. allows investors access to a CAL with a higher slope.
3. increases the utility of any individuals (in the absence of transaction costs).
4. Example 2 (cont): Ignoring DP. The slope of the CAL available using all 8 assets is higher than that for the CAL available using only the 5 individual stocks.

Efficient Frontier for the 5 Stocks and for the 8 Assets with and without T-bills: Ignoring DP
\( \{\sigma[R_i], E[R_i]\} \)s marked by \( \times \) (stocks) and +
C. Transaction Costs.

1. When the transaction costs associated with forming portfolios increase with the number of assets in the portfolio, there may be some optimal number of assets to have in the portfolio.

2. In this case, assets are added to the portfolio until the benefits from adding one more asset are offset by the associated increase in transactions costs.

3. Example 2 (cont): Ignoring DP. If an investor has used the 3 funds and T-bills to form a portfolio, the benefit from adding the 5 individual stocks appears small (see the graph below). If the investor faces significant fixed costs to start trading individual stocks (find a broker, open a brokerage account, ...) then the individual may prefer not to trade individual stocks.