Lecture 7: Bond Pricing, Forward Rates and the Yield Curve.

I. Reading.
II. Discount Bond Yields and Prices.
III. Fixed-income Prices and No Arbitrage.
IV. The Yield Curve.
V. Other Bond Pricing Issues.
VI. Holding Period Return.
VII. Forward Rates.
VIII. Theories of the Yield Curve.
Lecture 7: Bond Pricing, Forward Rates and the Yield Curve.

I. Reading.
A. BKM, Chapter 15.

II. Discount Bond Yields and Prices.
A. Relation between Prices and Yields for Discount Bonds.
   1. Yields are usually quoted in the industry as APRs with semiannual compounding; i.e., as bond equivalent yields.
   2. Let \( p_{\tau}(t) \) be the price at time \( t \) on a \( \tau \)-year discount bond with face value \( C_\tau(t+\tau) \).
   3. For discount bonds, yield (expressed as an APR with semiannual compounding) is related to price in the following way:

   \[
   p_{\tau}(t) = \frac{C_\tau(t+\tau)}{[1 + \frac{y_\tau(t)}{2}]^{2\tau}} \Rightarrow y_\tau(t) = 2 \left\{ \frac{C_\tau(t+\tau)}{p_\tau(t)} \right\}^{1/(2\tau)} - 1 \quad .
   \]

   4. Example:
      a. Government note and strip prices for 2/15/95.
      b. One period is a year.

Government Bonds and Notes.

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Ask Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Aug 95</td>
<td>?</td>
</tr>
<tr>
<td>4</td>
<td>Feb 96</td>
<td>?</td>
</tr>
<tr>
<td>6</td>
<td>Aug 96</td>
<td>?</td>
</tr>
</tbody>
</table>

U.S. Treasury Strips.

<table>
<thead>
<tr>
<th>Type</th>
<th>Maturity</th>
<th>Ask Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>ci</td>
<td>Aug 95</td>
<td>97</td>
</tr>
<tr>
<td>ci</td>
<td>Feb 96</td>
<td>94</td>
</tr>
<tr>
<td>ci</td>
<td>Aug 96</td>
<td>90</td>
</tr>
</tbody>
</table>

c. Can calculate the yield on a six month discount bond (expressed as an APR with semi-annual compounding) using the price of the
Aug 95 strip:

\[ y_{\frac{1}{2}} \text{ (Feb 95)} = 2 \times \{\frac{100}{97} - 1\} = 6.186\% . \]

d. Can calculate the yield on a one year discount bond (expressed as an APR with semi-annual compounding) using the price of the Feb 96 strip:

\[ y_1 \text{ (Feb 95)} = 2 \times \{\left[\frac{100}{94}\right]^\frac{1}{2} - 1\} = 6.284\% . \]

e. Can calculate the yield on a 1.5 year discount bond (expressed as an APR with semi-annual compounding) using the price of the Aug 96 strip:

\[ y_{1\frac{1}{2}} \text{ (Feb 95)} = 2 \times \{\left[\frac{100}{90}\right]^\frac{1}{3} - 1\} = 7.149\% . \]

B. Yields on Discount Bonds expressed as Discount Factors.

1. The \( \tau \)-period discount factor at time \( t \), denoted \( d_\tau(t) \), is the price at \( t \) of $1 received for certain at time \( t+\tau \).

2. The \( \tau \)-period discount factor at time \( t \) is analogous to the PVIF discussed in the time value of money.

\[
d_\tau(t) = \frac{1}{\left[1 + \frac{y_\tau(t)}{2}\right]^{2\tau}} \quad \Rightarrow \quad y_\tau(t) = 2 \left\{ \left[ \frac{1}{d_\tau(t)} \right]^{1/(2\tau)} - 1 \right\} .
\]

3. These formulas are used below to calculate the yield on a discount bond when the relevant discount factor is known and to calculate the relevant discount factor when the discount bond’s yield is known.

4. Example (cont):

a. Can calculate the \( \frac{1}{2} \) year discount factor on 2/15/95 using the yield of the Aug 95 strip (expressed as an APR with semi-annual compounding):

\[
d_{\frac{1}{2}}(Feb\ 95) = \frac{1}{\left[1 + \frac{0.06186}{2}\right]^1} = 0.97.
\]

b. Can calculate the 1 year discount factor on 2/15/95 using the yield of the Feb 96 strip (expressed as an APR with semi-annual compounding):
d_{1}(Feb 95) = \frac{1}{\left[1 + \frac{0.06284}{2}\right]^2} = 0.94.

c. Can calculate the 1\frac{1}{2} year discount factor on 2/15/95 using the yield of the Aug 96 strip (expressed as an APR with semi-annual compounding):

d_{1\frac{1}{2}}(Feb 95) = \frac{1}{\left[1 + \frac{0.07149}{2}\right]^3} = 0.90.

5. If the price of a discount bond paying C_{t}(t+\tau) in \tau periods is p_{t}(t), then d_{\tau}(t) is given by p_{t}(t)/C_{t}(t+\tau).
   a. Can calculate d_{\frac{1}{2}}(Feb 95) as follows using the Aug 95 strip:
   d_{\frac{1}{2}}(Feb 95) = p_{\frac{1}{2}}(Feb 95)/100 = 97/100 = 0.97.
   b. Can calculate d_{1}(Feb 95) as follows using the Feb 96 strip:
   d_{1}(Feb 95) = p_{1}(Feb 95)/100 = 94/100 = 0.94.
   c. Can calculate d_{1\frac{1}{2}}(Feb 95) as follows using the Aug 96 strip:
   d_{1\frac{1}{2}}(Feb 95) = p_{1\frac{1}{2}}(Feb 95)/100 = 90/100 = 0.90.

III. Fixed-income Prices and No Arbitrage
   A. An Arbitrage Opportunity.
      1. Definition: An investment that does not require any cash outflows and generates a strictly positive cash inflow with some probability is known as an arbitrage opportunity.
      2. In well functioning markets arbitrage opportunities can not exist since any individual who prefers more to less wants to invest as much as possible in the arbitrage opportunity.

      1. The absence of arbitrage implies that any two assets with the same stream of riskless cash flows must have the same price.
      2. This implication is known as the law of one price.
      3. Otherwise, could buy the lower priced asset and sell the higher priced asset and earn an arbitrage profit.
         a. Zero cash flows in the future.
         b. Positive cash flow today.
      4. Example: WSJ for 2/18/97. Compare U.S. Treasury Strips maturing on

1. Any bond $i$ paying the certain cash flow stream, $C_i(t+\frac{1}{2})$, $C_i(t+1)$, ..., $C_i(t+N)$, must have the following price at time $t$ for there to be no arbitrage:

$$P_i(t) = d_{\frac{1}{2}}(t) C_i(t+\frac{1}{2}) + d_1(t) C_i(t+1) + \ldots + d_N(t) C_i(t+N).$$

2. Alternatively, the price of any bond can be obtained by discounting back each certain cash flow at the yield on the discount bond with the same maturity as the cash flow:

$$P_i(t) = \frac{1}{\left[1 + \frac{y_{\frac{1}{2}}(0)}{2}\right]^1} C_i(t+\frac{1}{2}) + \frac{1}{\left[1 + \frac{y_1(0)}{2}\right]^2} C_i(t+1) + \ldots + \frac{1}{\left[1 + \frac{y_N(0)}{2}\right]^{2N}} C_i(t+N).$$

3. Example (cont): Since above we calculated the discount factors at Feb 95 for Aug 95, Feb 96 and Aug 96, we can determine the value each of the three bonds/notes in the absence of arbitrage using this formula:

a. 4 Aug 95:

$$P_{4\text{Aug 95}}(\text{Feb 95}) = d_{\frac{1}{2}}(\text{Feb 95}) \times \left[100 + \frac{4}{2}\right] = 0.97 \times 102 = 98.94.$$

b. 4 Feb 96:

$$P_{4\text{Feb 96}}(\text{Feb 95}) = d_{\frac{1}{2}}(\text{Feb 95}) \times \left[\frac{4}{2}\right] + d_1(\text{Feb 95}) \times \left[100 + \frac{4}{2}\right] = 0.97 \times 2 + 0.94 \times 102 = 97.82.$$

c. 6 Aug 96:

$$P_{6\text{Aug 96}}(\text{Feb 95}) = d_{\frac{1}{2}}(\text{Feb 95}) \times \left[\frac{6}{2}\right] + d_1(\text{Feb 95}) \times \left[6/2\right] + d_{\frac{1}{2}}(\text{Feb 95}) \times \left[100 + \frac{6}{2}\right] = 0.97 \times 3 + 0.94 \times 3 + 0.90 \times 103 = 98.43.$$
D. How to Earn an Arbitrage Profit when the Law of One Price is Violated.

1. Basic Idea.
   a. Buy the undervalued assets and sell the overvalued assets.
   b. Need to choose the weights to ensure that all cash flows are zero or positive.
   c. Easiest way is to choose the weights so that all future cash flows are zero; then today’s cash flow will be positive if there is an arbitrage opportunity.

2. How to use coupon bonds and a mispriced strip to create an arbitrage position.
   a. Example (cont): Suppose the 4 Feb 96 note is priced at 98 on 2/15/95 which is too high relative to the prices of the Aug 95 and Feb 96 strips.
      (1) Must sell the 4 Feb 96 note.
      (2) The idea is to
         (a) sell the 4 Feb 96 note (overpriced); and
         (b) buy a “synthetic” 4 Feb 96 note created using the Aug 95 and Feb 96 strips.
      (3) Must buy the Aug 95 and Feb 96 strips.
      (4) More specifically:
         (a) let \( a \) be the number of Feb 96 strips bought.
         (b) let \( b \) be the number of Aug 95 strips bought.

<table>
<thead>
<tr>
<th>Position</th>
<th>2/15/95</th>
<th>8/15/95</th>
<th>2/15/96</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell 1 4 Feb 96 note</td>
<td>SELL</td>
<td>1 x 98</td>
<td>-1 x 4/2</td>
</tr>
<tr>
<td>Note</td>
<td>NOTE</td>
<td>= 98</td>
<td>= -2</td>
</tr>
<tr>
<td>Buy ( a ) Feb 96 strips</td>
<td>BUY</td>
<td>-a x 94</td>
<td></td>
</tr>
<tr>
<td>SYNTETHIC NOTE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy ( b ) Aug 95 strips</td>
<td>BUY</td>
<td>-b x 97</td>
<td></td>
</tr>
</tbody>
</table>

| Net                    | 98 - a 94 - b 97 | 0 | 0 |

(5) So \( a \) 100 = 102 implies \( a = 1.02 \).
(6) So \( b \) 100 = 2 implies \( b = 0.02 \).
(7) His position earns an arbitrage profit of:

\[
98 - 1.02 \times 94 - 0.02 \times 97 = 0.18 \text{ today.}
\]

(8) It seems like a lot of trouble for 18 cents. But sell 1M of the 4 Feb 96 note and the profit becomes $180000.
IV. The Yield Curve.
A. Yield Curve Definition.
1. Discount bonds of differing maturities can have different yields to maturity.
2. The yield curve is just the yield to maturity (YTM) on a n-year discount bond graphed as a function of n.
3. Note that the YTM on the shortest maturity discount bond of interest is known as the spot rate.
4. It is not correct to use the yield to maturity on a n-year coupon bond as the yield on a n-year zero coupon bond; these are not the same.
5. Example: 2/15/95
   a. Yield on a 1½ year discount bond (expressed as an APR with semi-annual compounding) is 7.149%.
   b. Price of the 6 Aug 96 note can be calculated using law of one price:

   \[
P_{6 \text{ Aug } 96 (\text{Feb } 95)} = \frac{1}{[1 + \frac{0.06186}{2}]^1} + \frac{1}{[1 + \frac{0.06284}{2}]^2} + \frac{1}{[1 + \frac{0.07149}{2}]^3} = 103 \]

   which is lower than the yield on a 1½ year discount bond.

d. The reason is that the note has coupon cash flows prior to Aug 96 and the yields on ½ year and 1 year discount bonds are less than 7.149%.

e. YTM on the 10 Aug 96 note is 7.106% which is even lower than for the 6 Aug 96 note since a greater portion on the 10 Aug 96's cash flows occur prior to Aug 96 than the 6 Aug 96 note.

   c. YTM on the 6 Aug 96 note is 7.122% since

   \[
   98.43 = \frac{1}{[1 + \frac{0.07122}{2}]^1} + \frac{1}{[1 + \frac{0.07122}{2}]^2} + \frac{1}{[1 + \frac{0.07122}{2}]^3} = 103.
   \]

   which is lower than the yield on a 1½ year discount bond.
B. Yield Curve Calculation.
1. Short end can be obtained from U.S. T-bill yields.
2. Long end can be obtained from U.S. Treasury strips.
3. It is also possible to obtain the yield curve using U.S. Treasury notes and bonds despite the presence of coupon payments.
   a. It is not correct to use the yield to maturity on a n-year coupon bond as the yield on a n-year zero coupon bond: these are not the same.
   b. The concept of no arbitrage in well functioning markets can be used.
4. Generally, to replicate a T-period discount bond using coupon bonds:
   a. Use the T-period coupon bond to generate the time T cash flow; so buy the T-period coupon bond.
   b. Use the (T-1) period bond to offset the (T-1)th coupon from the T-period coupon bond; so sell the (T-1)-period coupon bond.
   c. Use the (T-2)-period bond to offset the sum of time (T-2) coupons for the T-period and (T-1)-period coupon bond.
   d. Continue until the cash flows for all points in time between now and time T are zero.
5. Once a discount bond has been replicated, its price and cashflow can be used to calculate its yield.
6. Example (cont): 2/15/95

**Government Bonds and Notes.**

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Ask Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Aug 95</td>
<td>98.94</td>
</tr>
<tr>
<td>4</td>
<td>Feb 96</td>
<td>97.82</td>
</tr>
<tr>
<td>6</td>
<td>Aug 96</td>
<td>98.43</td>
</tr>
</tbody>
</table>

a. Can recover the six month discount factor at 2/15/95 using the Aug 95 note since it has only one payment left on 8/15/95:

\[ d_{\frac{1}{2}} \text{ (Feb 95)} = \frac{98.94}{100 + \frac{4}{2}} = 0.97. \]

b. Can recover the 1 year discount factor at 2/15/95 bond the 4 Feb 96 notes and \( d_{\frac{1}{2}} \text{ (Feb 95)} \):

\[
P^{4 \text{ Feb 96}} \text{(Feb 95)} = 2 \ d_{\frac{1}{2}} \text{ (Feb 95)} + 102 = 2 \times 0.97 + 102 \ d_{1} \text{ (Feb 95)} \Rightarrow d_{1} \text{ (Feb 95)} = 0.94.
\]

c. Alternatively, can replicate a 1 year discount bond (with par value=100) at 2/15/95 using the 4 Aug 95 and the 4 Feb 96 notes:

(1) More specifically:

(a) let \( a \) be the number of 4 Feb 96 notes bought.

(b) let \( b \) be the number of 4 Aug 95 notes bought.

<table>
<thead>
<tr>
<th>Position</th>
<th>2/15/95</th>
<th>8/15/95</th>
<th>2/15/96</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy ( a ) 4 Feb 96 notes</td>
<td>BUY SYNTHETIC DISCOUNT BOND</td>
<td>(-a \times 97.82)</td>
<td>(a \times 102)</td>
</tr>
<tr>
<td>Buy ( b ) 4 Aug 95 notes</td>
<td></td>
<td>(-b \times 98.94)</td>
<td>(b \times 102)</td>
</tr>
<tr>
<td>Net</td>
<td>(-a \times 97.82 - b \times 98.94)</td>
<td>0</td>
<td>100</td>
</tr>
</tbody>
</table>

(2) So \( a \times 102 = 100 \) implies \( a = 0.9804 \).

(3) So \( a + b \times 102 = 0 \) implies \( b = -0.0192 \). Since \( b \) is negative, this means that strategy involves selling 0.0192 of the 4 Aug 95 notes.

(4) Thus, the cost of the synthetic discount bond is

\[ 0.9804 \times 97.82 - 0.0192 \times 98.94 = 94 \Rightarrow d_{1} \text{ (Feb 95)} = 94/100 = 0.94. \]
V. Other Bond Pricing Issues.

A. Introduction.
1. Sometimes prices appear to violate the law of one price.
2. Usually there is a reason for an apparent violation.
3. Many fixed income traders are constantly looking for arbitrage opportunities. They are in a position to act on any such opportunity immediately. Thus, it is unlikely that an arbitrage opportunity could be identified by looking at the WSJ.

B. Reasons for deviations from the law of one price.
1. Taxes.
   a. If two fixed income instruments have the same pretax cash flows but these cash flows are taxed differently, then the two instruments may have different prices
2. Liquidity.
   a. If the liquidity of two fixed income instruments differ, so too may their prices.
3. Transaction Costs.
   a. When there are transaction costs in the form of a bid ask spread, no arbitrage implies the ask price for a given instrument must be greater than some cutoff and its bid price must be less than another cutoff.

C. Default Risk.
1. Fixed income instruments with the same promised cash flows may have different prices if they have differing probabilities of default.
2. If a fixed income instrument has some probability of default:
   a. expected cash flows from the instrument are less than the promised cash flows.
   b. there is uncertainty or risk associated with the cash flows; so investors require a higher expected return to hold a corporate compared to an equivalent treasury.
3. Hence, the yield on a discount bond with default risk (e.g., corporate debt) will be greater than the yield on a discount bond of the same maturity without any default risk (e.g., government bills).
VI. Holding Period Return.
   A. Definition.
      1. Let $h_i(t+\frac{1}{2})$ denote the $\frac{1}{2}$ year holding period return on bond $i$ over the $\frac{1}{2}$ year ending at $t+\frac{1}{2}$.
      2. The $\frac{1}{2}$ year holding period return is defined as follows:

\[
h(t) = \frac{p(t+\frac{1}{2}) + C(t+\frac{1}{2}) - p(t)}{p(t)}
\]

where
- $p(t)$ is the price of the bond at time $t$; if the bond matures at time $t$, $p(t)$ is the face value of the bond;
- $C(t)$ is the coupon payment at time $t$ on the bond.

3. The holding period return on a bond is analogous to the return on an asset (which has been defined already).

<table>
<thead>
<tr>
<th>1 Period Strip</th>
<th>Treasury</th>
<th>Corporate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Promised Payment $P$</td>
<td>110</td>
<td>110</td>
</tr>
<tr>
<td>Prob of Default</td>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>Expected Payment $E[CF]$</td>
<td>110</td>
<td>55</td>
</tr>
<tr>
<td>Expected Return $E[R]$</td>
<td>10%</td>
<td>10%</td>
</tr>
<tr>
<td>Price</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>YTM</td>
<td>10%</td>
<td>120%</td>
</tr>
</tbody>
</table>
4. Example: Government strip, note and bond prices 2/15/95

**Government Bonds and Notes.**

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Ask Price</th>
<th>Ask Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>Aug 96</td>
<td>98.43</td>
<td>7.122%</td>
</tr>
</tbody>
</table>

**U.S. Treasury Strips.**

<table>
<thead>
<tr>
<th>Type</th>
<th>Maturity</th>
<th>Ask Price</th>
<th>Ask Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>ci</td>
<td>Aug 95</td>
<td>97</td>
<td>6.186%</td>
</tr>
<tr>
<td>ci</td>
<td>Feb 96</td>
<td>94</td>
<td>6.284%</td>
</tr>
<tr>
<td>ci</td>
<td>Aug 96</td>
<td>90</td>
<td>7.149%</td>
</tr>
</tbody>
</table>

**Government Bonds and Notes.**

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Ask Price</th>
<th>Ask Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>Aug 96</td>
<td>97.7</td>
<td>8.446%</td>
</tr>
</tbody>
</table>

**U.S. Treasury Strips.**

<table>
<thead>
<tr>
<th>Type</th>
<th>Maturity</th>
<th>Ask Price</th>
<th>Ask Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>ci</td>
<td>Aug 95</td>
<td>100</td>
<td>-</td>
</tr>
<tr>
<td>ci</td>
<td>Feb 96</td>
<td>98</td>
<td>4.082%</td>
</tr>
<tr>
<td>ci</td>
<td>Aug 96</td>
<td>92</td>
<td>8.514%</td>
</tr>
</tbody>
</table>
a. What is the 6 month holding period return on Aug 95 strips from 2/15/95 to 8/15/95?

\[ h_{\text{Aug 95 strip}}(8/15/97) = \frac{(100-97)}{97} = 3.093\% \]

(which is 2 \times 3.093\% = 6.186\% when expressed as an APR with semi-annual compounding).

b. What is the 6 month holding period return on Feb 96 strips from 2/15/95 to 8/15/95?

\[ h_{\text{Feb 96 strip}}(8/15/97) = \frac{(98-94)}{94} = 4.255\% \]

(which is 2 \times 4.255\% = 8.510\% when expressed as an APR with semi-annual compounding).

c. What is the 6 month holding period return on the 6 Aug 96 notes from 2/15/95 to 8/15/95?

\[ h_{\text{6 Aug 96 note}}(8/15/97) = \frac{(97.7+3-98.43)}{98.43} = 2.306\% \]

(which is 2 \times 2.306\% = 4.612\% when expressed as an APR with semi-annual compounding).

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Holding Period Return (2/15/95-8/15/95)</th>
<th>YTM</th>
</tr>
</thead>
<tbody>
<tr>
<td>6% Aug 96 Note</td>
<td>4.612%</td>
<td>7.122%</td>
</tr>
<tr>
<td>Feb 96 Strip</td>
<td>8.510%</td>
<td>6.284%</td>
</tr>
</tbody>
</table>

d. Sensitivity Analysis for the 6% Aug 96 Note.

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Holding Period Return (2/15/95-8/15/95)</th>
<th>(2/15/95)</th>
<th>(8/15/95)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>YTM</td>
<td>Price</td>
<td>YTM</td>
</tr>
<tr>
<td>6% Aug 96 Note</td>
<td>4.612%</td>
<td>7.122%</td>
<td>98.43</td>
</tr>
<tr>
<td>6% Aug 96 Note</td>
<td>7.122%</td>
<td>7.122%</td>
<td>98.43</td>
</tr>
<tr>
<td>6% Aug 96 Note</td>
<td>8.317%</td>
<td>7.122%</td>
<td>98.43</td>
</tr>
</tbody>
</table>
B. Note.

1. The ½ year holding period return on a ½ year discount bond equals its yield to maturity.
   a. Example. YTM on Aug 95 strip at 2/15/95 is 6.186% which equals its holding period return (expressed as an APR with 6 month compounding) over the ½ year period 2/15/95 to 8/15/95.

2. For any bond of longer maturity, its ½ year holding period return need not equal either its own yield to maturity or the YTM on a ½ year discount bond.
   a. Example. Holding period return (expressed as an APR with 6 month compounding) of the 6 Aug 96 note is 4.612% over the ½ year period 2/15/95 to 8/15/95. YTM on Aug 95 strip at 2/15/95 is 6.186% while the YTM on the 6 Aug 96 note at 2/15/95 is 7.122%.

3. For any bond, if its YTM at the end of the holding period is greater than its YTM at the start of the holding period, then its holding period return will be less than its YTM at the start of the holding period.
   a. Example. YTM on the 6 Aug 96 note at 8/15/95 is 8.446% which is greater than its YTM at 2/15/95 of 7.122%. Thus, its holding period return (expressed as an APR with 6 month compounding) over the ½ year period 2/15/95 to 8/15/95 is 4.612% which is less than its YTM as of 2/15/95 of 7.122%.

4. For any bond, if its YTM at the end of the holding period is less than its YTM at the start of the holding period, then its holding period return will be greater than its YTM at the start of the holding period.
   a. Example. YTM on the Feb 96 strip at 8/15/95 is 4.082% which is less than its YTM at 2/15/95 of 6.284%. Thus, its holding period return (expressed as an APR with 6 month compounding) over the ½ year period 2/15/95 to 8/15/95 is 8.51% which is greater than its YTM as of 2/15/95 of 6.284%.

5. When a bond pays coupons within the holding period, there is a question how these should be incorporated into the holding period return calculation.
VII. Forward Rates.

A. Definition.

1. Example: It is 2/15/95. You want to lock in an interest rate for the period 8/15/95 to 2/15/96. So agree today to invest an amount on 8/15/95 for 6 months at an interest rate (expressed as an APR with semi-annual compounding) of 6.38%.
   a. This rate is known as the 6 month forward rate available on 2/15/95 for the period starting on 8/15/95.
   b. So if I agree on 2/15/95 to invest $1 on 8/15/95, I will receive $1 + 0.0638/2 = $1.0319 on 2/15/96.

2. Let $f_{t,t+\alpha}(0)$ be the $\alpha$-period forward rate (expressed as an APR with semiannual compounding) available at time 0 for the period starting $t$ periods after time 0.
   a. Example. You want to lock in an interest rate for the period 8/15/95 to 2/15/96. The relevant forward rate is $f_{8/15,1}(Feb\ 95)$.
   b. If an investor knew at time 0 that she would have $1 M available to invest at time $t$ and wanted a certain rate of return on a $\alpha$ period investment, then $f_{t,t+\alpha}(0)$ is the applicable yield expressed as an APR with semiannual compounding.

3. Why is there a need for forward contracts?
   a. Because the yield curve shifts through time in a manner that is not totally predictable.
   b. Example.
      (1) Using a forward contract on 2/15/95, an investor can lock in an APR of 6.38% over the period 8/15/95 to 2/15/96.
      (2) If the investor waits until 8/15/95 to invest for the six month period, she may get a higher or lower rate.
      (3) Given the yield curve on 8/15/95 above, the investor would only have received an APR of 4.082% (YTM on the Feb 96 strip as of 8/15/95).
      (4) Investor may not want this uncertainty.
B. Forward rates expressed as forward contract discount factors.

1. Let $d_{t,t+\alpha}(0)$ be the price paid at $t$ for $1$ delivered at $(t+\alpha)$ which can be locked in at time 0.

   a. Example. You want to lock in an interest rate for the period 8/15/95 to 2/15/96. The relevant discount factor is $d_{\frac{8}{15},1}(Feb\ 95)$.

   b. Now $d_{t,t+\alpha}(0)$ is referred to as the $\alpha$-period forward contract discount factor available at time 0 for the period starting at time $t$.

2. It is possible to go from $d_{t,t+\alpha}(0)$ to $f_{t,t+\alpha}(0)$ and vice versa using analogous formulas to those for discount bonds:

$$d_{t,t+\alpha}(0) = \frac{1}{\left[1 + \frac{f_{t,t+\alpha}(0)}{2}\right]^{2\alpha}} \implies f_{t,t+\alpha}(0) = 2 \left\{ \frac{1}{d_{t,t+\alpha}(0)} \right\}^{1/(2\alpha)} - 1 \right\}.$$

   a. Example (cont):

   (1) Can calculate the discount factor associated with the forward rate available on 2/15/95 for the period from 8/15/95 to 2/15/96 using the forward rate $f_{\frac{8}{15},1}(Feb\ 95) = 6.38\%$:

$$d_{\frac{8}{15},1}(Feb\ 95) = \frac{1}{\left[1 + \frac{0.0638}{2}\right]} = 0.9691.$$

   (2) Can calculate the forward rate (expressed as an APR with semi-annual compounding) available on 2/15/95 for the period from 8/15/95 to 2/15/96 using the relevant discount factor for that forward rate $d_{\frac{8}{15},1}(Feb\ 95) = 0.9691$:

$$f_{\frac{8}{15},1}(Feb\ 95) = 2 \times \{1/0.9691 - 1\} = 6.38\%.$$
C. Relation between Forward Rates and Discount Bond Yields.

1. Example (cont):
   a. On 2/15/95, we can use Aug 95 discount bonds and the 6 month forward rate available for the period starting on 8/15/95 to replicate a Feb 96 discount bond.
   b. Thus, should be able to use the discount bond yields $y_{95}$ (Feb 95) = 6.186%; and $y_{1}$ (Feb 95) = 6.284%.
      and the law of one price to get the forward rate $f_{95,1}$ (Feb 95).
   c. What are the cash flows?

<table>
<thead>
<tr>
<th>Date</th>
<th>2/15/95</th>
<th>8/15/95</th>
<th>2/15/96</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
</tr>
</tbody>
</table>

   BUY $1 worth of Feb 96 strips
   -1 $[1+ \frac{y_{1}(0)}{2}]^2$ Feb 96 strips

   BUY $1 worth of Aug 95 strips and roll proceeds into a 6 month forward contract for the period starting 8/15/95
   -1 $[1+ \frac{y_{95}(0)}{2}]$ Aug 95 strips
   -1 $[1+ \frac{y_{95}(0)}{2}][1+ f_{95,1}(0)/2]$ forward contract
   -1 $[1+ \frac{y_{95}(0)}{2}][1+ f_{95,1}(0)/2]$ forward contract

   d. Since both investments generate a certain cash flow on 2/15/96, the law of one price implies that

   $[1+ \frac{y_{1}(0)}{2}]^2 = [1+ \frac{y_{95}(0)}{2}][1+ f_{95,1}(0)/2]$

   $[1+ 0.06284/2]^2 = [1+ 0.06186/2][1+ f_{95,1}(0)/2] \Rightarrow f_{95,1}(0) = 6.38\%.$

2. In general,

   $[1 + \frac{y_{t+a}(0)}{2}]^{2(t+a)} = [1 + \frac{y_{t}(0)}{2}]^{2t} [1 + \frac{f_{t+t+a}(0)}{2}]^{2a}$

   which implies

   $d_{t+a}(0) = d_{t}(0) d_{t+a}(0)$.

3. Example. What is the 1-year forward rate expressed as an APR with
semianual compounding available on 2/15/95 for the period starting 8/15/95?
a. First calculate \( d_{\frac{1}{2},1\frac{1}{2}} \) (Feb 95). Using the formula,

\[
d_{\frac{1}{2},1\frac{1}{2}} \text{ (Feb 95)} = d_{1\frac{1}{2}} \text{ (Feb 95)} / d_{\frac{1}{2}} \text{ (Feb 95)} = 0.90 / 0.97 = 0.9278.
\]

b. Then get the forward rate \( f_{\frac{1}{2},1\frac{1}{2}} \) (Feb 95) as follows:

\[
f_{\frac{1}{2},1\frac{1}{2}} \text{ (Feb 95)} = 2 \times \{ (1/0.9278)^{\frac{1}{2}} - 1 \} = 7.63\%.
\]

4. Example. Bloomberg screen. 4/1/97. The yield on a 3-year discount bond \( y_3(0) \) is 6.56%. The 1-year forward rate for the year starting in 3 years \( f_{3,4}(0) \) is 6.90%. The yield on a 4-year discount bond \( y_4(0) \) can be obtained as follows:

a. First, use above formula

\[
\left[ 1 + \frac{y_4(0)}{2} \right]^{2 \times 4} = \left[ 1 + \frac{y_3(0)}{2} \right]^{2 \times 3} \left[ 1 + \frac{f_{3,4}(0)}{2} \right]^{2 \times 1}
\]

\[
= \left[ 1 + \frac{0.0656}{2} \right]^6 \left[ 1 + \frac{0.069}{2} \right]^2 = 1.2988
\]

b. Then obtain \( y_4(0) \):

\[
y_4(0) = 2 \times \left\{ 1.2988^{\frac{1}{8}} - 1 \right\} = 6.64\%
\]

which agrees with the Bloomberg screen.
VIII. Theories of the Yield Curve.

A. Forward Rates determine the Yield Curve.

1. The relation between forward rates and the yield on discount bonds can be specialized to the following:

\[
[1 + \frac{y_{t+\frac{1}{2}}(0)}{2}]^{2(t+\frac{1}{2})} = [1 + \frac{y_t(0)}{2}]^{2t} \left[1 + \frac{f_{t,t+\frac{1}{2}}(0)}{2}\right].
\]

2. It follows that the shape of the yield curve at any point in time depends on the spot rate \(y_{\frac{1}{2}}(0)\) together with the sequence of \(\frac{1}{2}\) year forward rates available at that time.
   a. If \(f_{t,t+\frac{1}{2}}(0) > y_t(0)\) then \(y_{t+\frac{1}{2}}(0) > y_t(0)\).
   b. If \(f_{t,t+\frac{1}{2}}(0) < y_t(0)\) then \(y_{t+\frac{1}{2}}(0) < y_t(0)\).

3. Note:
   a. an upward sloping yield curve implies that the sequence of \(\frac{1}{2}\) year forward rates are higher than the current spot rate \(y_{\frac{1}{2}}(0)\).
   b. a downward sloping yield curve implies that the sequence of \(\frac{1}{2}\) year forward rates are all lower than the current spot rate \(y_{\frac{1}{2}}(0)\).
B. How are forward rates determined?
   1. Expectations Theory.
      a. According to this theory, today’s forward rate for any future ½ year period is equal to the expected interest rate over that ½ year period.
      b. So
         \[ f_{t,t+\frac{1}{2}}(0) = E_{at \text{ time } 0} [y_{\frac{1}{2}}(t)]. \]
      c. Idea is that if an investor does not care about uncertainty then the expected payoff at time \((t+\frac{1}{2})\) from the following two strategies should be the same:
         (1) Buy $1 of t-year discount bonds and commit to invest the proceeds for ½ a year at \(f_{t,t+\frac{1}{2}}(0)\): the certain payoff is
            \[
            \left[ 1 + \frac{y_t(0)}{2} \right]^2 \left[ 1 + \frac{f_{t,t+\frac{1}{2}}(0)}{2} \right]^1.
            \]
         (2) Buy $1 of t-year discount bonds and invest the proceeds for a ½ year at the prevailing spot rate at time \(t\) which will be \(y_{\frac{1}{2}}(t)\): expected payoff is
            \[
            \left[ 1 + \frac{y_t(0)}{2} \right]^2 \left[ 1 + \frac{E_{at \text{ time } 0} [y_{\frac{1}{2}}(t)]}{2} \right]^1.
            \]
      d. This argument is not an arbitrage argument; it relies on individuals not caring about the uncertainty associated with the second strategy.
      e. If the expectations hypothesis holds, then:
         (1) an upward sloping yield curve implies that the market is expecting higher spot rates in the future.
         (2) a downward sloping yield curve implies that the market is expecting lower spot rates in the future.
Lecture 7

2. Liquidity Preference Theory.
   a. This theory says that forward rates embody a liquidity premia in addition to the expected future spot rate.
   b. So

   \[ f_{t,t+\frac{1}{2}}(0) = E_{\text{at time } 0}[y_{t}(t)] + p_{t+\frac{1}{2}}(0) \]

   where \( p_{t+\frac{1}{2}}(0) \) is the liquidity premium.
   c. Theory says that:
      (1) \( p_{t+\frac{1}{2}}(0) > 0 \) for all \( t>0 \).
   d. The idea is that individuals require a higher return to induce them to invest in less liquid, longer maturity bonds.
   e. If the liquidity preference hypothesis holds, then:
      (1) an upward sloping yield curve is consistent with the market expecting higher or lower spot rates in the future.
      (2) a downward sloping yield curve implies that the market is expecting lower spot rates in the future.

3. Preferred Habitat Theory.
   a. This theory says that yields on different securities are determined by the supply and demand for that security and that securities with similar maturities are not close substitutes.
   b. Theory says that:
      (1) \( p_{t+\frac{1}{2}}(0) \) can be positive or negative for a given \( t \) depending on supply and demand for the \((t+\frac{1}{2})\)-period discount bond.
   c. If the preferred habitat hypothesis holds, then the slope of the yield curve tell you little about whether the market is expecting higher or lower spot rates in the future.
\[
\left[1 + \frac{y_{t+\frac{1}{2}}(0)}{2}\right]^{2(t+\frac{1}{2})} = \left[1 + \frac{y_t(0)}{2}\right]^{2t} \left[1 + \frac{f_{t,t+\frac{1}{2}}(0)}{2}\right]^t.
\]

\[
\left[1 + \frac{y_{\frac{3}{2}}(0)}{2}\right] \cdot
\]

\[
\left[1 + \frac{y_1(0)}{2}\right]^2 = \left[1 + \frac{y_{\frac{3}{2}}(0)}{2}\right] \left[1 + \frac{f_{\frac{3}{2},1}(0)}{2}\right].
\]

\[
\left[1 + \frac{y_{1+\frac{1}{2}}(0)}{2}\right]^3 = \left[1 + \frac{y_1(0)}{2}\right]^2 \left[1 + \frac{f_{1,1+\frac{1}{2}}(0)}{2}\right].
\]

\[
\left[1 + \frac{y_2(0)}{2}\right]^4 = \left[1 + \frac{y_{1+\frac{1}{2}}(0)}{2}\right]^3 \left[1 + \frac{f_{1,2}(0)}{2}\right].
\]

\[
\left[1 + \frac{y_{2+\frac{1}{2}}(0)}{2}\right]^5 = \left[1 + \frac{y_2(0)}{2}\right]^4 \left[1 + \frac{f_{2,2+\frac{1}{2}}(0)}{2}\right].
\]

...