Lecture 9: Options: Valuation.

I. Reading.

II. Preliminaries.

III. No Arbitrage Pricing Bounds.
   A. Call Options.
   B. Put Options.
   C. Put Call Parity.

IV. Black Scholes Model.
   A. Assumptions.
   B. Notation.
   C. Formula for European Call Options.
   D. Value of an American call option.
   E. European Put Options.
   F. American Put Options.
   G. Implied Volatility.
I. Reading.
   A. BKM, Chapter 20, Section 20.4.
   B. BKM, Chapter 21, ignore Section 21.3 and skim Section 21.5.

II. Preliminaries.
   A. Up until now, we have been concerned with the payoffs of put and call options at maturity. This handout is concerned with:
      1. The value of a call or put option prior to maturity.
      2. Whether an American call option or an American put option should be exercised prior to maturity.
   B. The results in this handout refer to non-dividend paying underlying assets unless otherwise stated.
   C. Notation.
      1. $S(0)$ be the value of the underlying at time 0.
      2. $d_T(0)$ be the discount factor for a $T$-year discount bond available at time 0.
      3. $C_{X,T}(0)$ be the time 0 price of a European call option with exercise price $X$ and expiration date $T$.
      4. $c_{X,T}(0)$ be the time 0 price of an American call option with exercise price $X$ and expiration date $T$.
      5. $P_{X,T}(0)$ be the time 0 price of a European put option with exercise price $X$ and expiration date $T$.
      6. $p_{X,T}(0)$ be the time 0 price of an American put option with exercise price $X$ and expiration date $T$. 
III. No Arbitrage Pricing Bounds.

A. Call Options.

1. Floor on the Value of a European Call.
   a. Have already established that a European call option must have a nonnegative price since its cashflow at expiration is nonnegative: $C_{X,T}(0) \geq 0$.
   b. Consider the following two investment strategies:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S(T)&lt;50$</td>
<td>$S(T) \geq 50$</td>
</tr>
<tr>
<td>Buy underlying at 0 and sell at T</td>
<td>-$S(0)$</td>
<td>$S(T)$</td>
</tr>
<tr>
<td>Sell a T period discount bond with face</td>
<td>50 $d_1(0)$</td>
<td>-50</td>
</tr>
<tr>
<td>value of 50 and hold to maturity</td>
<td></td>
<td>-50</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>-$S(0) + 50 d_1(0)$</td>
<td>$S(T)-50$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S(T)&lt;50$</td>
<td>$S(T) \geq 50$</td>
</tr>
<tr>
<td>Buy a European call option at 0 with</td>
<td>-$C_{50,T}(0)$</td>
<td>0</td>
</tr>
<tr>
<td>exercise price of 50 and hold until</td>
<td></td>
<td></td>
</tr>
<tr>
<td>expiration at T</td>
<td></td>
<td>$S(T)-50$</td>
</tr>
</tbody>
</table>
c. Can see that the second strategy always produces a cash flow equal to or greater than the first strategy:

1. If $S(T) \geq 50$, both strategies generate the same cash flow at $T$: $[S(T)-50]$.
2. If $S(T) < 50$, the first strategy generates $[S(T)-50]$ while the second strategy generates 0 which is $>[S(T)-50]$.

d. For there not to exist an arbitrage opportunity, the second strategy must cost more than the first one; i.e., $C_{50,T}(0) \geq S(0)-50 d_t(0)$. This restriction is a floor on the call’s value.

e. So more generally, $C_{X,T}(0) \geq S(0) - X d_t(0)$ and $C_{X,T}(0) \geq 0$ which implies that $C_{X,T}(0) \geq \max\{S(0) - X d_t(0), 0\}$. 

![Call on Stock expiring at T with $50$ exercise price: Floor on Call Price at T](image)
Example. Bloomberg screen for 4/15/97 Microsoft options. May 97 options expire 5/17/97. Microsoft is not paying any dividend between 4/15/96 and 5/17/96. The “Fin Rate” is the riskfree rate and is 5.28%. Assuming this is a continuously compounded annual rate, it implies a discount factor on a 32-day discount bond of $e^{0.0528 \times \frac{32}{365}} = 0.9954$.

\[ \text{(1) For } X=85, \]
\[ \max\{S(0) - X, 0\} = \max\{98.375 - 85 \times 0.9954, 0\} = \max\{13.766, 0\} = 13.766 < 14.5 \]

which is the ask price of the call.

\[ \text{(2) Suppose the ask price of the 85 call on 4/15/97 is 13.5 which violates the floor of 13.766. There is an arbitrage opportunity which involves buying the call (since it is undervalued) and selling the first strategy described above:} \]

<table>
<thead>
<tr>
<th>Strategy</th>
<th>4/15/97</th>
<th>5/17/97</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 1 Msft call option on 4/15/97 with exercise price of 85 and hold until expiration on 5/17/97</td>
<td>-13.5</td>
<td>0</td>
</tr>
<tr>
<td>Sell 1 Msft share on 4/15/97 and close out on 5/17/97</td>
<td>98.375</td>
<td>-S(5/17/97)</td>
</tr>
<tr>
<td>On 4/15/97, buy a discount bond with face value of 85 maturing on 5/17/97 and hold to maturity</td>
<td>-85 \times 0.9954 = -84.609</td>
<td>85</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>0.266</td>
<td>85-S(5/17/97)&gt;0</td>
</tr>
</tbody>
</table>
2. Early Exercise of an American Call.
   a. Know that an American option must be worth at least as much as a European option with the same expiry date and exercise price:
      \[ c_{X,T}(0) \geq C_{X,T}(0). \]
   b. So the American call has the same floor as the European call.
   c. The value of a European option with same expiry date and exercise price can be thought of as the value of holding the American option til the exercise date.
   d. If this value exceeds the value of exercising the American option now, and this is true for any date prior to \( T \), then it is optimal to hold the American option til maturity.
   e. In general, the floor on the value of the call associated with not exercising prior to maturity is always greater than or equal to the value of exercising the call now (since \( d_T(0) < 1 \)):
      \[ c_{X,T}(0) \geq \max\{S(0) - X d_T(0), 0\} \geq \max\{S(0) - X, 0\}. \]
   f. So the holder of an American call option on a non-dividend paying underlying never wants to exercise early.
**B. Put Options.**

1. **Floor on the Value of a European Put.**
   
a. Consider the following two investment strategies:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell underlying at 0 and close out at T</td>
<td>S(0)</td>
<td>-S(T)</td>
</tr>
<tr>
<td>Buy a T period discount bond with face value</td>
<td>-50 d_r(0)</td>
<td>50</td>
</tr>
<tr>
<td>of 50 and hold to maturity</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>S(0) -50 d_r(0)</td>
<td>50 -S(T)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy a European put option at 0 with</td>
<td>-P_{50,T}(0)</td>
<td>[50-S(T)]</td>
</tr>
<tr>
<td>exercise price of 50 and hold until</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>expiration at T</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Lecture 9  

b. Can see that the second strategy always produces a cash flow equal to or greater than the first strategy:
   (1) If \( S(T) < 50 \), both strategies generate the same cash flow at \( T: [50-S(T)] \).
   (2) If \( S(T) \geq 50 \), the first strategy generates \([50-S(T)]\) while the second strategy generates 0 which is \( \geq [50-S(T)] \).

c. For there not to exist an arbitrage opportunity, the second strategy must cost more than the first one; i.e., \( P_{50,T}(0) \geq 50 \, d_T(0) - S(0) \).

d. Combining this floor with the nonnegativity restriction, obtain:
   \[ P_{50,T}(0) \geq \max\{50 \, d_T(0) - S(0), 0\} \).

e. For general exercise price \( X \), obtain the following floor:
   \[ P_{X,T}(0) \geq \max\{X \, d_T(0) - S(0), 0\} \).

   (1) For \( X=85 \),
   \[ \max\{X \, d_T(0) - S(0), 0\} = \max\{85 \times 0.9954-98.5, 0\} = \max\{-13.891, 0\} = 0 < 0.875 \]
   which is the ask price of the put.
   (2) For \( X=105 \),
   \[ \max\{X \, d_T(0) - S(0), 0\} = \max\{105 \times 0.9954- 98.5, 0\} = \max\{6.017, 0\} = 6.017 < 8.5 \]
   which is the ask price of the put.

![Diagram](https://via.placeholder.com/150)
2. Early Exercise of an American Put.
   a. Know that an American option must be worth at least as much as a European option with the same expiry date and exercise price:
   \[ P_{X,T}(0) \geq P_{X,T}(0). \]
   b. The value of a European option with same expiry date and exercise price can be thought of as the value of holding the American option til the exercise date.
   c. In general, the floor on the value of the put associated with not exercising prior to maturity is less than the value of exercising the put now (since \( d_T(0) < 1 \)):
   \[ P_{X,T}(0) \geq \max \{ X - S(0), 0 \} \geq \max \{ X d_T(0) - S(0), 0 \}. \]
   d. So the holder of an American put option on a non-dividend paying underlying may want to exercise early.
   e. In fact, it can be shown that for \( S(0) \) sufficiently small, the holder of an American put on a non-dividend paying underlying prefers to exercise immediately.
C. Put Call Parity.

1. Consider the following two investment strategies:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S(T)&lt;50</td>
<td>S(T)&gt;50</td>
</tr>
<tr>
<td>Buy underlying at 0 and sell at T</td>
<td>-S(0)</td>
<td>S(T)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S(T)&lt;50</td>
<td>S(T)&gt;50</td>
</tr>
<tr>
<td>Buy a European call option at 0 with</td>
<td>-C_{50,T}(0)</td>
<td>0</td>
</tr>
<tr>
<td>exercise price of 50 and hold until</td>
<td></td>
<td>S(T)-50</td>
</tr>
<tr>
<td>expiration at T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Write a European put option at 0 with</td>
<td>P_{50,T}(0)</td>
<td>-[50-S(T)]</td>
</tr>
<tr>
<td>exercise price of 50 and hold until</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>expiration at T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy a T period discount bond with face</td>
<td>-50 d_{t}(0)</td>
<td>50</td>
</tr>
<tr>
<td>value of 50 and hold to maturity</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>-C_{50,T}(0)+P_{50,T}(0)</td>
<td>S(T)</td>
</tr>
<tr>
<td></td>
<td>-50 d_{t}(0)</td>
<td>S(T)</td>
</tr>
</tbody>
</table>

2. Can see that these two strategies have the same cash flows. The law of one price says that these strategies must have the same price.

3. Thus, get a relation between the price of a European call and put with the same exercise date and price and the price of the underlying and the present value of the exercise price:

\[ S(0) = C_{50,T}(0) - P_{50,T}(0) + 50 d_{t}(0). \]

4. For general exercise price X,

\[ S(0) = C_{X,T}(0) - P_{X,T}(0) + X d_{t}(0). \]

5. This relation is known as put call parity.

6. Can also see the relation using payoff diagrams. Sum the payoff diagrams for the second strategy and you get the payoff at T from holding the underlying.

7. Example. Msft price today is 98.385. The price of a discount bond (face value of 100) maturing in 32 days is 99.54. A European call expiring in 32 days with an exercise price of 85 has a price of 14.5 today. What is the price today of a European put expiring in 32 days with an exercise price of 85?

\[
P_{85,32\text{day}}(0) = C_{85,32\text{day}}(0) - S(0) + 85 d_{32\text{day}}(0) = 14.5 -98.375 +85 \times 0.9954 = 0.734.
\]
Long 1 European Call on Stock expiring at T with $50 exercise price: Payoff at T

Short 1 European Put on Stock expiring at T with $50 exercise price: Payoff at T

Long 1 European Call expiring at T with X=$50 and Short 1 European Put expiring at T with X=$50: Payoff at T
IV. Black Scholes Model.
A. Assumptions.
1. Yield curve is flat through time at the same interest rate. So there is no interest rate uncertainty
2. Underlying asset return is lognormally distributed with constant volatility and does not pay dividends.
3. Continuous trading is possible.
4. No transaction costs, taxes or other market imperfections.

B. Notation. Above notation holds. Additionally,
1. \( r' \) is the continuously compounded annual riskfree rate. So \$1 invested today at the riskless rate is worth \$1 \( e^{r't} \) in \( t \) years time.
2. \( \sigma \) is the volatility of the continuously compounded annual return on the underlying asset.

C. Formula for European Call Options.
1. The value of a European call option is given by:
\[
C_{X,T}(0) = S(0) \, N(d_1) - X \, e^{-r'T} \, N(d_2)
\]
where \( N(.) \) is the cumulative Normal distribution function (see BKM, Table 20.2);
\[
d_1 = \frac{\ln[S(0)/X] + \{r' + \sigma^2/2\} \, T}{\sigma \sqrt{T}}; \quad \text{and,} \quad d_2 = d_1 - \sigma \sqrt{T}.
\]
2. Factors affecting the value of the call.
   a. \( S(0) \): \( C_{X,T}(0) \) is monotonically increasing in \( S(0) \) as would be expected.
   b. \( X \): \( C_{X,T}(0) \) is monotonically decreasing in \( X \) as would be expected.
   c. \( \sigma \): \( C_{X,T}(0) \) is monotonically increasing in \( \sigma \). Why?
      (1) Option feature of the call truncates the payoff at 0 when the underlying’s value is less than the strike price.
      (2) When \( \sigma \) increases, the volatility of \( S(T) \) increases.
      (3) The call option holder benefits from the greater upside potential of \( S(T) \) but does not bear the greater downside potential due to the truncation of the option payoff at 0.
      (4) Thus, the value of the call increases relative to \( S(0) \).
   d. \( T \): \( C_{X,T}(0) \) is monotonically increasing in \( T \). Why?
      (1) The exercise price does not have to be paid until time \( T \). When \( T \) increases, the current value of \( X \) paid at \( T \) decreases making the option more valuable for given \( S(0) \).
      (2) Second, with a longer time to maturity the volatility of \( S(T) \) increases for given \( \sigma \). So the value of the call today increases for the same reason that an increase in \( \sigma \) increases the call’s value today.
      (3) Both effects are acting in the same direction.
   e. \( r' \): \( C_{X,T}(0) \) is monotonically increasing in \( r' \). Why?
      (1) The exercise price does not have to be paid until time \( T \). When \( r' \) increases, the current value of \( X \) paid at \( T \) decreases making the option more valuable for given \( S(0) \).

3. Factors not affecting the value of the call.
   a. the expected return on the underlying asset. Why?
      (1) When the expected return on the underlying increases, the expected return on the call also increases.
      (2) Since the current underlying’s price \( S(0) \) remains equal to the current value of the underlying despite its higher expected return, \( C(0) \) remains the current value of the call option despite its higher expected return.
D. Value of an American call option.
1. It was shown above that the holder of an American call option on a non-dividend paying asset would never exercise early.
2. Thus, the value of an American call equals the value of the European call with the same exercise price and date.
3. So the value of an American call is also given by the Black Scholes call option formula.
E. European Put Options.
1. Once the value of the European call with same exercise price and date has been determined, put call parity can be used to determine the value of a European put.
2. Factors affecting the value of the European put.
   a. $S(0)$: $P_{X,T}(0)$ is monotonically decreasing in $S(0)$ as would be expected.
   b. $X$: $P_{X,T}(0)$ is monotonically increasing in $X$ as would be expected.
   c. $\sigma$: $P_{X,T}(0)$ is monotonically increasing in $\sigma$. Why?
      (1) Option feature of the put truncates the payoff at 0 when the underlying’s value is more than the strike price.
      (2) When $\sigma$ increases, the volatility of $S(T)$ increases.
      (3) The put option holder benefits from the greater downside potential of $S(T)$ but does not bear the greater upside potential due to the truncation of the option payoff at 0.
      (4) Thus, the value of the put increases relative to $S(0)$.
   d. $T$: affect on $P_{X,T}(0)$ is ambiguous. Why?
      (1) The exercise price is not received until time $T$. When $T$ increases, the current value of $X$ received at $T$ decreases making the option less valuable for given $S(0)$.
      (2) But acting in the other direction, a longer time to maturity increases the volatility of $S(T)$ for a given $\sigma$. When $T$ increases, the value of the put today also increases for the same reason that an increase in $\sigma$ (for fixed $T$) increases the put’s value today.
      (3) It is not clear which of these effects dominates.
   e. $r'$: $P_{X,T}(0)$ is monotonically decreasing in $r'$. Why?
      (1) The exercise price is not received until time $T$. When $r'$ increases, the current value of $X$ received at $T$ decreases making the option less valuable for given $S(0)$.

F. American Put Options.
1. No closed form solution exists to the value of an American put option.
2. A value can be obtained numerically.

G. Implied Volatility.
1. All the inputs into the Black-Scholes model are readily observable except the volatility of the return on the underlying.
2. The value of $\sigma$ which together with the other Black-Scholes inputs gives a Black-Scholes call price equal to the market’s prevailing call price is known as the implied volatility of the underlying.
### Lecture 9: Foundations of Finance

#### States are Equally Likely

<table>
<thead>
<tr>
<th></th>
<th>Stock</th>
<th>Call $C_{50,T}$</th>
<th>Put $P_{50,T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff at $T$ - State 1</td>
<td>40</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>Payoff at $T$ - State 2</td>
<td>60</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$E[\text{Payoff at } T]$</td>
<td>50</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$\sigma[\text{Payoff at } T]$</td>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### States are Equally Likely

<table>
<thead>
<tr>
<th></th>
<th>Stock</th>
<th>Call $C_{50,T}$</th>
<th>Put $P_{50,T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff at $T$ - State 1</td>
<td>20</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>Payoff at $T$ - State 2</td>
<td>80</td>
<td>30</td>
<td>0</td>
</tr>
<tr>
<td>$E[\text{Payoff at } T]$</td>
<td>50</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>$\sigma[\text{Payoff at } T]$</td>
<td>30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### States are Equally Likely

<table>
<thead>
<tr>
<th></th>
<th>Stock</th>
<th>Call $C_{50,T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff at $T$ - State 1</td>
<td>20</td>
<td>25</td>
</tr>
<tr>
<td>Payoff at $T$ - State 2</td>
<td>80</td>
<td>85</td>
</tr>
<tr>
<td>$E[\text{Payoff at } T]$</td>
<td>50</td>
<td>55</td>
</tr>
<tr>
<td>$\sigma[\text{Payoff at } T]$</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Price(0)</td>
<td>40</td>
<td>10</td>
</tr>
<tr>
<td>$E[\text{Return}]$</td>
<td>25%</td>
<td>37.5%</td>
</tr>
<tr>
<td>States are Equally Likely</td>
<td>Stock S(t)</td>
<td>Call C_{50,T(t)}</td>
</tr>
<tr>
<td>---------------------------</td>
<td>------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>Payoff at T - State 1</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>Payoff at T - State 2</td>
<td>80</td>
<td>30</td>
</tr>
<tr>
<td>E[Payoff at T]</td>
<td>50</td>
<td>15</td>
</tr>
<tr>
<td>σ[Payoff at T]</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Price(0)</td>
<td>40</td>
<td>38</td>
</tr>
<tr>
<td>E[Return]</td>
<td>25%</td>
<td>31.6%</td>
</tr>
</tbody>
</table>
Lecture 9: Futures and Forward Contracts: Valuation.

I. Reading.
II. Futures Prices.
   A. Applicability of Spot Forward Parity.
III. Forward Prices: Spot Forward Parity.
   A. Introduction.
   C. General Case: Carrying Costs.
   D. Application to Foreign Currency Forward Contracts: Covered Interest Parity.
   E. Spot Forward Parity and Arbitrage.
Lecture 9: Futures and Forward Contracts: Valuation.

I. Reading.
   A. BKM, Chapter 22, Sections 22.4.
   B. BKM, Chapter 23, omit Section 23.5.

II. Futures Prices.
   A. Applicability of Spot Forward Parity.
      1. In general, the futures price need not equal the forward price and so spot forward parity need not hold exactly for futures contracts.
      2. However, spot forward parity can be expected to hold approximately for futures.

III. Forward Prices: Spot Forward Parity.
   A. Introduction.
      1. Interested in determining how the forward price is determined.
      2. Turns out that no arbitrage implies that the forward price must be related to the spot price in a very particular way.

<table>
<thead>
<tr>
<th>Underlying</th>
<th>Spot</th>
<th>Forward - deliver on 12/97</th>
<th>Growth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gold-1oz</td>
<td>339.9</td>
<td>349.6</td>
<td>2.9%</td>
</tr>
<tr>
<td>Cotton-100 lb</td>
<td>70.53</td>
<td>75.96</td>
<td>7.7%</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>775.2</td>
<td>791.55</td>
<td>2.1%</td>
</tr>
</tbody>
</table>

Why the differences?

1. Example 1:
   a. Consider a forward contract to deliver 1 oz of gold on 4/98 entered into on the 4/97. The spot price for 1 oz of gold is $400. The price of a U.S. T-bill ($100 face value) maturing on 4/98 is 80.
   b. What is the relation between the forward price and the spot price?
   c. Can replicate a forward contract by going long the stock and shorting discount bonds that mature on the settlement date with a face value equal to the forward price.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>4/97</th>
<th>4/98</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy a forward contract on 4/97 which delivers 1 oz of gold on 4/98</td>
<td>0</td>
<td>S(4/98) - F1(4/97)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>4/97</th>
<th>4/98</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 1 oz of gold on 4/97 and sell on 4/98</td>
<td>-400</td>
<td>S(4/98)</td>
</tr>
<tr>
<td>Sell 1-yr U.S. T-bills on 4/97 with face value of F1(4/97)</td>
<td>F1(4/97) 0.8</td>
<td>-F1(4/97)</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>F1(4/97) 0.8 -400</td>
<td>S(4/98) - F1(4/97)</td>
</tr>
</tbody>
</table>

d. Can see that these two strategies have the same cash flows.
e. By definition, no money changes hands today under the forward contract; so the law of one price says

\[ F_1(4/97) - 400 = 0; \]

\[ F_1(4/97) = 400 / 0.8 = 500. \]
f. Note:

1. if \( F_1(4/97) - 400 > 0 \) (i.e., \( F_1(4/97) > 500 \)), the buyer of the forward contract must be paid money today to be induced to enter the contract.

2. if \( F_1(4/97) - 400 < 0 \) (i.e., \( F_1(4/97) < 500 \)), the buyer of the forward contract would pay money today to be allowed to enter the contract.
2. Thus, have shown that the forward price is just the future value of the spot price at the settlement date (invested today in a discount bond maturing at the settlement date):

\[ F_T(0) = \frac{S(0)}{d_T(0)} \quad \text{or} \quad F_T(0) = S(0) [1 + y^*_T(0)]^T \]

where

a. \( T \) is time to settlement.

b. \( S(0) \) be the value of the underlying at time 0.

c. \( d_T(0) \) be the price of a \( T \) period discount bond with a $1 face value.

d. \( y^*_T(0) \) be the effective 1-period yield on a \( T \) period discount bond.

e. \( F_T(0) \) be the time 0 forward price of the underlying for delivery in \( T \) periods.

3. So at any point in time expect the forward price of gold to increase as the settlement date becomes more distant since the future value of the spot price increases with maturity.
C. General Case: Carrying Costs.

1. Suppose there are certain costs associated with holding or carrying an asset between time 0 and the settlement of the forward contract at time T.
   a. Example: When a physical commodity is the underlying, any storage costs are a carrying cost.
   b. Example: If the underlying is a stock index, dividend payments by the stocks in the index are a negative carrying cost.

2. Assume the carrying costs are known at time 0.

3. Example 2:
   a. Consider a forward contract to deliver 100 lb of cotton on 4/98 entered into on the 4/97. The spot price on 4/97 for 100 lb of cotton is $71. The price of a U.S. T-bill ($100 face value) maturing on 4/98 is 80. The cost of storing 100 lb of cotton from 4/97 to 4/98 is $10 payable on 10/97. The price of a U.S. T-bill ($100 face value) maturing on 10/97 is 90.

   b. What is the relation between the forward price and the spot price?

   c. Consider the following two investment strategies:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>4/97</th>
<th>10/97</th>
<th>4/98</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy a forward contract on 4/97 which delivers 100 lb cotton on 4/98</td>
<td>0</td>
<td>0</td>
<td>$S(4/98) - F_{f}(4/97)$</td>
</tr>
<tr>
<td>Buy 100 lb cotton on 4/97 and sell on 4/98</td>
<td>-71</td>
<td>-10</td>
<td>$S(4/98)$</td>
</tr>
<tr>
<td>Buy a ½-year discount bond on 4/97 with face value of 10 and close out at maturity</td>
<td>-10 x 0.9</td>
<td>10</td>
<td>-</td>
</tr>
<tr>
<td>Sell a 1 year discount bond on 4/97 with face value of $F_{f}(4/97)$ and hold to maturity</td>
<td>$F_{f}(4/97) 0.8$</td>
<td></td>
<td>$-F_{f}(4/97)$</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>$F_{f}(4/97) 0.8 - 10x 0.9 - 71$</td>
<td>0</td>
<td>$S(4/98) - F_{f}(4/97)$</td>
</tr>
</tbody>
</table>
d. Can see that these two strategies have the same cash flows.
e. By definition, no money changes hands today under the forward contract; so the law of one price says:

\[ 0 = F_1(4/97) \times 0.8 - 10 \times 0.9 - 71 \]

f. So

\[ F_1(4/97) = (71 + 9)/0.8 = 100. \]

4. Thus, get a relation between the current spot price and the forward price:

\[ F_T(0) d_T(0) - C(t_c) d_{tc}(0) = S(0) \]

or

\[ F_T(0) \frac{1}{[1+y^*(0)]^T} - C(t_c) \frac{1}{[1+y^*_c(0)]^c} = S(0) \]

where:

a. the carrying costs \( C(t_c) \) are paid at time \( t_c \) between times 0 and T (usually will take the settlement date T to be the date at which the costs are paid).
b. \( S(0) \) be the value of the underlying at time 0.
c. \( d_T(0) \) be the price of a \( \tau \) period discount bond with a $1 face value.
d. \( y^*_c(0) \) be the effective 1-period yield on a \( \tau \) period discount bond.
e. \( F_T(0) \) be the time 0 forward price of the underlying for delivery in T periods.

5. This relation is known as spot forward parity and can be rewritten:

\[ F_T(0) = [1+y^*_T(0)]^{T} [ S(0) + C(t_c) \frac{1}{[1+y^*_c(0)]^c} ] \]

6. Can see that

a. a positive carrying cost implies a higher forward price.
b. a negative carrying cost (e.g., dividend-paying underlying) implies a lower forward price.

7. Example 3:

a. The S&P 500 index is 800 on 4/97. The price of a discount bond (face value of 100) maturing on 4/98 is 80. The stocks in the index will pay dividends amounting to 40 of index value on 10/97. The price of a discount bond (face value of 100) maturing on 10/97 is 90. What is forward price on 4/97 for delivery of the index on 4/98?

\[ F_1(4/97) 0.8 - (-40) 0.9 = 800 \rightarrow F_1(4/97) = (800 -36)/0.8 = 955. \]
D. Application to Foreign Currency Forward Contracts: Covered Interest Parity.

1. In the case of foreign currency forward contracts, spot forward parity is known as covered interest parity.

2. For a forward contract to deliver a foreign currency in $T$ years, the underlying is the foreign currency. The negative carrying cost is the interest received from investing the foreign currency in a discount bond maturing in $T$ years.

3. Example 4:
   a. The spot price for a British pound on 4/97 is $1.60$: i.e., $S_{£}(4/97) = 1.60$. The yield on a 1 year discount bond denominated in U.S. dollars is 25% while the yield on a 1 year discount bond denominated in pounds is 10%. What is the forward price on 4/97 for delivery of one £ on 4/98?
   b. Consider the following two strategies:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>4/97</th>
<th>4/98</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy $1 of 1-year $-denominated discount bonds on 4/97 and sell on 4/98</td>
<td>-1</td>
<td>1 + 0.25</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell a forward contract which delivers $[1/1.6][1+ 0.1] = 0.6875 £ on 4/98</td>
<td>0</td>
<td>${F_{1.6}(4/97)-S_{£}(4/98)} \times [1/1.6][1+0.1] = F_{1.6}(4/97) [1/1.6][1+0.1] - S_{£}(4/98) \times [1/1.6][1+0.1]$</td>
</tr>
<tr>
<td>Buy $1 worth of £ on 4/97 (£1/1.6) and invest in 1-year £-denominated discount bonds and hold till maturity. On 4/98, convert the $[1/1.6][1+ 0.1] = 0.6875 £$ to $$$ at $S_{£}(4/98)$</td>
<td>-1</td>
<td>$S_{£}(4/98)[1/1.6][1+0.1]$</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td></td>
<td>$F_{1.6}(4/97) [1/1.6][1+0.1]$</td>
</tr>
</tbody>
</table>

c. The first strategy is buying a 1-year $-denominated discount bond while the second is creating a synthetic 1-year $-denominated discount bond on 4/97 by:
   (1) buying pounds on 4/97,
   (2) investing the proceeds in 1-year £-denominated discount bonds, and
   (3) locking in on 4/97 the exchange rate (the forward rate) at which the pounds can be converted back to dollars on 4/98.
d. Can see that these two strategies cost $1 on 4/97 and generate a certain dollar cash flow on 4/98. The law of one price says that the certain dollar cash flows on 4/98 must be the same:

\[ 1 + 0.25 = F_{1}^{\$/\text{£}(4/97)} \left[ 1/1.6 \right] [1+0.1] \]

and so \( F_{1}^{\$/\text{£}(4/97)} = 1.8181/\text{£} \).

4. Thus obtain the following result which is the covered interest parity theorem:

\[ [1+y_{T}(0)]^{T} = [1+y_{T}(0)]^{T} F_{T}^{\$/\text{£}(0)/S^{\$/\text{£}(0)}} \]

where

a. \( y_{T}(0) \) is the effective per period yield on a T period discount bond denominated in U.S. dollars.

b. \( y_{T}(0) \) is the effective per period yield on a T period discount bond denominated in £.

c. \( S^{\$/\text{£}(0)} \) is the spot price of 1 £ at time 0.

d. \( F_{T}^{\$/\text{£}(0)} \) be the forward price at time 0 for 1 £ delivered in T periods.

5. Can see that:

a. if the yield on the foreign currency discount bond is lower than on the dollar-denominated discount bond, the forward price of the foreign currency (in $s) is higher that the spot price (in $s).

b. if the yield on the foreign currency discount bond is higher than on the dollar-denominated discount bond, the forward price of the foreign currency (in $s) is lower that the spot price (in $s).
**E. Spot Forward Parity and Arbitrage.**

1. If spot forward parity is violated, there is an arbitrage opportunity.

2. Example 1 (cont):
   a. Suppose the forward price on 4/97 for delivery of 1 oz of gold on 4/98 is 520 which is greater than the 500 implied by spot forward parity.
   b. So the forward price is too high which implies that you want to sell forward contracts and buy the underlying:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>4/97</th>
<th>4/98</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell a forward contract on 4/97 which delivers 1 oz of gold on 4/98</td>
<td>0</td>
<td>520 - $\text{S}(4/98)$</td>
</tr>
<tr>
<td>Buy 1 oz of gold on 4/97 and sell on 4/98</td>
<td>-400</td>
<td>$\text{S}(4/98)$</td>
</tr>
<tr>
<td>Sell 1-yr U.S. T-bills on 4/97 with face value of 520</td>
<td>520 x 0.8</td>
<td>-520</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>16</td>
<td>0</td>
</tr>
</tbody>
</table>

c. This strategy is an arbitrage opportunity.

3. Example 4 (cont):
   a. Suppose the forward price on 4/97 for delivery of a £ on 4/98 is $2 which is higher than $1.8181 implied by covered interest parity.
   b. Want to buy the synthetic 1-year $-denominated discount bond and sell the 1-year $-denominated discount bond:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>4/97</th>
<th>4/98</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell $1 of 1-year $-denominated discount bonds on 4/97 and close out on 4/98</td>
<td>1</td>
<td>-1.25</td>
</tr>
<tr>
<td>Sell a forward contract which delivers $\frac{1}{1.6}[1 + 0.1] = 0.6875$ £ on 4/98</td>
<td>0</td>
<td>$[2 - S^{\text{f}}(4/98)] \times 0.6875 = 2 \times 0.6875 - S^{\text{f}}(4/98) \times 0.6875$</td>
</tr>
<tr>
<td>Buy $1 worth of £ on 4/97 (£1/1.6) and invest in 1-year £-denominated discount bonds and hold til maturity. On 4/28, convert the $[1/1.6][1 + 0.1] = 0.6875$ £ to $ at S^{\text{f}}(4/98).$</td>
<td>-1</td>
<td>$S^{\text{f}}(4/98) \times 0.6875$</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>0</td>
<td>2 \times 0.6875 - 1.25 = 0.125</td>
</tr>
</tbody>
</table>

c. This strategy is an arbitrage opportunity.