Problem Set 1 Solution: Time Value of Money and Equity Markets

I. **Present Value with Multiple Cash Flows:**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A:</td>
<td>40000</td>
<td>40000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B:</td>
<td>30000</td>
<td>20000</td>
<td>20000</td>
<td></td>
</tr>
</tbody>
</table>

APR is 16% compounded quarterly; Periodic Rate (with quarterly compounding) is \( r_{1/4} = 0.16/4 = 0.04 = 4\% \). EAR is \((1+r_{1/4})^4 - 1 = 1.04^4 - 1 = 0.16986 = 16.986\% \).

Salary Arrangement A:

\[
V^A_0 = $40000 \times PVAF_{16.986\%,2} = $40000 \times \frac{1-(1.16986)^2}{0.16986} = $63419.66.
\]

Salary Arrangement B:

\[
V^B_0 = $30000 + $20000 \times PVAF_{16.986\%,2} = $30000 + $20000 \times \frac{1-(1.16986)^2}{0.16986} = $30000 + $31709.83 = $61709.83.
\]

II. **Calculating EAR and continuous compounding:**

<table>
<thead>
<tr>
<th>APR ((i_{\text{nom}}))</th>
<th>Compound Periods in a Year ((m))</th>
<th>Periodic Rate ((r_{1/m}))</th>
<th>EAR ((r))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>2</td>
<td>0.02</td>
<td>0.0404</td>
</tr>
<tr>
<td>0.06</td>
<td>4</td>
<td>0.015</td>
<td>0.061364</td>
</tr>
<tr>
<td>0.18</td>
<td>365</td>
<td>0.0004931</td>
<td>0.19716</td>
</tr>
</tbody>
</table>

When the continuously compounded APR (\(m=\infty\)) is 0.22, then \( \text{EAR} = r = \exp\{0.22\} - 1 = 0.24608 \) or 24.608\%.

III. **EAR vs APR:**

Let 1 period be a year. Question says that the effective monthly rate \( (r_{1/12}) \) is 0.2. So: APR \((i_{\text{nom}})\) is 12 \( \times \) 0.2\% = 2.4 or 240\%.

EAR \((r)\) is \((1+0.2)^{12} - 1 = 7.9161\) or 791.61\%.
IV.  Calculating Annuity Payments:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th>65</th>
<th>66</th>
<th>67</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>w:</td>
<td></td>
<td></td>
<td>10000</td>
<td>10000</td>
<td>10000</td>
<td></td>
</tr>
<tr>
<td>d:</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**A.**

Define $V_{w65}^w$ to be the single sum equivalent at age 65 of the desired withdrawal stream. Now using the present value annuity factor gives the single sum equivalent of the withdrawal annuity stream at age 65 (since the first withdrawal is made at age 66):

$$V_{w65}^w = \$10000 \times \frac{1 - (1.08)^{-10}}{0.08} = \$10000 \times 6.71008 = \$67100.8.$$  

Define $V_{d65}^d$ to be the single sum equivalent at age 65 of the necessary deposit stream. For the stream of deposits to be sufficient to allow the stream of withdrawals to be made:

$$V_{d65}^d = V_{w65}^w = \$67100.8$$

But using the future value annuity factor gives the single sum equivalent of the deposit annuity stream at age 65 (since the last deposit is made at age 65):

$$V_{d65}^d = \$67100.8 = D \times FVAF_{8\%,30} = D \times \frac{(1.08)^{30} - 1}{0.08} = D \times 113.28321$$

and so $D = \$67100.8/113.2832 = \$592.33$.

**B.**

Using the single sum present value formula,

$$V_{36}^d = V_{65}^d \times PVIF_{8\%,29} = \$67100.8 \times (1 + 0.08)^{-29} = \$7201.8.$$  

V.  Loan Amortization:  Consider a 20 year $90000 mortgage loan with monthly payments. Assume an APR of 9% compounded monthly. The $90000 is lent today and the first payment is in one month's time.

**A.**  What is the size of the monthly payments?
APR \((i_{\text{nom}})\) is 9% with monthly compounding. 
Period Rate \((r)\) is \(i_{\text{nom}}/12 = 9%/12 = 0.75\% \) which is the effective monthly rate.

The present value annuity factor gives the single sum equivalent at time 0 (since the first payment must be made in one month):
\[
V_{240}^0 = 90000 = C \times PVAF_{0.75\%,240} = C \times \left[\frac{1-(1.0075)^{-240}}{0.0075}\right] = C \times 111.14495
\]
and so \(C = \frac{90000}{111.14495} = 809.75336\).

B. What is the principal outstanding in 10 years from now (just after the 120th payment)?

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 119 & 120 & 121 & 240 \\
\hline
& & & & 809.75 & 809.75 \\
\end{array}
\]

The balance of the loan outstanding after the 120th payment is just the single sum equivalent at time 120 for the last 120 payments. The present value annuity factor gives the single sum equivalent at time 120 for the last 120 payments (since the first of these payments is made at time 121):
\[
V_{120}^{120} = 809.75336 \times PVAF_{0.75\%,120} = 809.75336 \times \left[\frac{1-(1.0075)^{-120}}{0.0075}\right] \\
= 809.75336 \times 78.9417 = 63923.301.
\]

C. What portion of the 120th payment goes to principal and what goes to interest?

To calculate the interest accruing on the loan during the 120th month, need to calculate the balance outstanding after the 119th payment.

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 119 & 120 & 121 & 240 \\
\hline
& & & & 809.75 & 809.75 & 809.75 \\
\end{array}
\]

As for the previous part, the principal outstanding after the 119th payment is just the single sum equivalent at time 119 for the last 121 payments. The present value annuity factor gives the single sum equivalent at time 119 for the last 121 payments (since the first of these payments is made at time 120):
\[
V_{119}^{121} = 809.75336 \times PVAF_{0.75\%,121} = 809.75336 \times \left[\frac{1-(1.0075)^{-121}}{0.0075}\right] \\
= 809.75336 \times 79.34659 = 64251.17.
\]

Interest = \(64251.17 \times 0.0075 = 481.8838\).
However, it does not matter how much of the 120th payment goes to interest. The balance of the loan still drops from $64251.17 after the 119th payment to $63923.30 after the 120th payment.

VI. **Deferred Annuities:** Consider a single premium deferred annuity (SPDA) which costs $28765.5 and promises yearly payments of $20000 every year beginning 21 years from now. If the advertised EAR for the SPDA is 9%, how many payments must the SPDA make? (use trial and error if you cannot solve it with algebra)

Let $N$ be the number of payments.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>...</th>
<th>20+N</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>20000</td>
<td>20000</td>
<td>20000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$28765.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$V_{20}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

EAR ($r$) is 9%.

Using the present value annuity factor,

$V_{20} = 20000 \times \text{PVAF}_{9\%,N} = 20000 \times \frac{1 - (1.09)^{-N}}{0.09}$.

Using the present value interest factor for single sums:

$28765.5 = V_0 = V_{20} \times \text{PVIF}_{9\%,20} = V_{20} \times (1.09)^{-20}$.

But then

$28765.5 = 20000 \times \frac{1 - (1.09)^{-N}}{0.09} \times (1.09)^{-20}$.

The rest is algebra:

$(28765.5/20000) \times (1.09)^{-20} = \frac{1 - (1.09)^{-N}}{0.09}$

and

$(28765.5/20000) \times 0.09 = 1 - (1.09)^{-N}$

and

$1 - (28765.5/20000) \times 0.09 = (1.09)^{-N}$

and

$\ln [1 - (28765.5/20000) \times 0.09] = \ln [(1.09)^{-N}]$

and

$\ln [1 - (28765.5/20000) \times 0.09] = -N \ln [(1.09)]$

and finally

$N = -\ln [(1 - (28765.5/20000) \times 0.09) / \ln [(1.09)] = -[-1.292664/0.0861776] = 15$. 

VII. The Limit-order Book of the NYSE Specialist: BKM, Chapter 3, Question 7, parts a. and b.
Part a. The market-buy order will be filled at the lowest limit-sell order which is $50.25.
Part b. The next market-buy order will be filled at the lowest remaining limit-sell order which is $51.50.

VIII. Buying Stock on Margin: BKM, Chapter 3, Question 15 (for part b. assume that interest on the broker’s loan does not accrue until the end of the year).
Part a.
A. One Answer.
1. You have $5000 today.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>Net Worth</td>
</tr>
<tr>
<td>5000</td>
<td>5000</td>
</tr>
<tr>
<td>Total Asset</td>
<td>Total Liab &amp; Net W.</td>
</tr>
<tr>
<td>5000</td>
<td>5000</td>
</tr>
</tbody>
</table>

2. You borrow $5000 today.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>Loan</td>
</tr>
<tr>
<td>10000</td>
<td>5000</td>
</tr>
<tr>
<td></td>
<td>Net Worth</td>
</tr>
<tr>
<td></td>
<td>5000</td>
</tr>
<tr>
<td>Total Asset</td>
<td>Total Liab &amp; Net W.</td>
</tr>
<tr>
<td>10000</td>
<td>10000</td>
</tr>
</tbody>
</table>

3. You invest the $10000 cash in 200 shares of AT&T. Percent Margin = $5000/10000 = 50%.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>AT@T (200 sh @50)</td>
<td>Loan</td>
</tr>
<tr>
<td>10000</td>
<td>5000</td>
</tr>
<tr>
<td></td>
<td>Net Worth</td>
</tr>
<tr>
<td></td>
<td>5000</td>
</tr>
<tr>
<td>Total Asset</td>
<td>Total Liab &amp; Net W.</td>
</tr>
<tr>
<td>10000</td>
<td>10000</td>
</tr>
</tbody>
</table>

4. At the end of the period, the value of AT&T shares grows 10% to $55 while the amount outstanding on the loan grows 8%. Percent Margin = \{11000-5400\}/11000 = 50.909%.
5. At the end of the period, close out the position.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>AT&amp;T (200sh @$55)</td>
<td>Loan (5000 x 1.08) 5400</td>
</tr>
<tr>
<td>Net Worth</td>
<td>5600</td>
</tr>
<tr>
<td>Total Asset 11000</td>
<td>Total Liab &amp; Net W. 11000</td>
</tr>
</tbody>
</table>

6. Your return is:

\[
\text{Return} = \frac{\text{Net Worth at End} - \text{Net Worth at Start}}{\text{Net Worth at Start}} = \frac{5600 - 5000}{5000} = 12\%.
\]

B. Second Answer.

At start, $10000 is invested in AT&T by buying 200 shares.

Profit
\[
= \text{Value of 200 AT&T shares at end} - \text{Loan Outstanding at end} - \text{Net Worth at Start}
\]
\[
= 200 \times ($50 \times 1.1) - ($5000 \times 1.08) - $5000
\]
\[
= $11000 - $5400 - $5000 = $600.
\]

Return = Profit/Net Worth at Start = $600/$5000 = 12%.

Part b. Know
Margin = (Value of Stock Position - Loan Outstanding)/ Value of Stock Position.

So
\[
0.30 = \frac{200 \times \text{Price of AT&T} - $5000}{200 \times \text{Price of AT&T}}
\]
\[
0.30 \times (200 \times \text{Price of AT&T} - $5000) = 140 \times \text{Price of AT&T} = $5000
\]

Price of AT&T = $35.71.

IX. Short-selling when the Stock Pays a Dividend: BKM, Chapter 3, Question 18 (assume the interest rate on any required margin for the short position is zero).

Assume that there is the margin requirement is 50%. For short sales,
Margin = Net Worth in Broker’s Account / Value of Stock.

So need to have a net balance in broker’s account of 0.5 x (100 x $14) = $700. So need to deposit $700 into the broker’s account on 1/1.

A. One Answer.

1. On 1/1, you borrow 100 shares of Zenith at $14.00 per share. The loan is
denominated in shares of IBM not in dollars.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 sh @ 14</td>
<td>Loan (100 sh @ 14) 1400</td>
</tr>
<tr>
<td>Cash</td>
<td>Net Worth 700</td>
</tr>
<tr>
<td>Total Asset</td>
<td>Total Liab &amp; Net W. 2100</td>
</tr>
</tbody>
</table>

2. You sell the borrowed shares at $14 per share on 1/1.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>Loan (100 sh @ 14) 1400</td>
</tr>
<tr>
<td>less Commission (100 sh @ 0.5)</td>
<td>Net Worth 650</td>
</tr>
<tr>
<td>Total Asset</td>
<td>Total Liab &amp; Net W. 2050</td>
</tr>
</tbody>
</table>

3. On the 3/1, Zenith pays a $2 dividend per share which you have to pay. (Assume that Zenith’s price is still $14 per share.)

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>Loan (100 sh @ 14) 1400</td>
</tr>
<tr>
<td>less Dividend (100 sh @ 2)</td>
<td>Net Worth 450</td>
</tr>
<tr>
<td>Total Asset</td>
<td>Total Liab &amp; Net W. 1850</td>
</tr>
</tbody>
</table>

4. On the 4/1, Zenith’s price is $9 per share..

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>Loan (100 sh @ 9) 900</td>
</tr>
<tr>
<td></td>
<td>Net Worth 950</td>
</tr>
<tr>
<td>Total Asset</td>
<td>Total Liab &amp; Net W. 1850</td>
</tr>
</tbody>
</table>
5. To close out the position on 4/1, take the following steps:

(1) purchase 100 shares of Zenith stock.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash 950</td>
<td>Loan (100 sh @ 9) 900</td>
</tr>
<tr>
<td>less Commission (100 sh @ 0.5) -50</td>
<td></td>
</tr>
<tr>
<td>100 sh @ 9 900</td>
<td>Net Worth 900</td>
</tr>
<tr>
<td>Total Asset 1800</td>
<td>Total Liab &amp; Net W. 1800</td>
</tr>
</tbody>
</table>

(2) repay the stock loan.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash 900</td>
<td>Net Worth 900</td>
</tr>
<tr>
<td>Total Asset 900</td>
<td>Total Liab &amp; Net W. 900</td>
</tr>
</tbody>
</table>

Profit = Net Worth (4/1) - Net Worth (1/1) = $900 - $700 = $200.

B. Second Answer.

Profit = Proceeds Sale of 100 sh at $14 on 1/1 - Cost of Purchasing 100 sh at $9 on 4/1
- Payment of Dividend of $2 per share on 100 sh - 2 x Commission on 100 sh at $0.50
= $1400 - $900 - $200 - 2x $50 = $200.
Problem Set Solutions  Foundations of Finance

Problem Set 2 Solution: Portfolio Management and the CAPM

I.  *Expected Return, Return Standard Deviation, Covariance and Portfolios:*

<table>
<thead>
<tr>
<th>State</th>
<th>Probability</th>
<th>Asset A</th>
<th>Asset B</th>
<th>Riskless Asset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boom</td>
<td>0.25</td>
<td>24%</td>
<td>14%</td>
<td>7%</td>
</tr>
<tr>
<td>Normal Growth</td>
<td>0.5</td>
<td>18%</td>
<td>9%</td>
<td>7%</td>
</tr>
<tr>
<td>Recession</td>
<td>0.25</td>
<td>2%</td>
<td>5%</td>
<td>7%</td>
</tr>
</tbody>
</table>

A. What is the expected return on each asset?

\[
E[R_A] = 0.25 \times 24\% + 0.5 \times 18\% + 0.25 \times 2\% = 15.5\%.
\]

\[
E[R_B] = 0.25 \times 14\% + 0.5 \times 9\% + 0.25 \times 5\% = 9.25\%.
\]

\[
E[R_f] = R_f = 0.25 \times 7\% + 0.5 \times 7\% + 0.25 \times 7\% = 7\%.
\]

B. What is the standard deviation of return on each asset?

First, calculate variance

\[
\sigma_{R_A}^2 = 0.25 \times (24\times24) + 0.5 \times (18\times18) + 0.25 \times (2\times2) - (15.5\times15.5) = 307 - 240.25 = 66.75.
\]

\[
\sigma_{R_B}^2 = 0.25 \times (14\times14) + 0.5 \times (9\times9) + 0.25 \times (5\times5) - (9.25\times9.25) = 95.75 - 85.5625 = 10.1875.
\]

\[
\sigma_{R_f}^2 = 0.25 \times (7\times7) + 0.5 \times (7\times7) + 0.25 \times (7\times7) - (7\times7) = 49 - 49 = 0.
\]

Then calculate standard deviation

\[
\sigma_{R_A} = 8.1701\%.
\]

\[
\sigma_{R_B} = 3.1918\%.
\]

\[
\sigma_{R_f} = 0\%.
\]

C. What is the correlation and covariance between the returns on

1. assets A and B?

Covariance:

\[
\sigma[R_A, R_B] = 0.25 \times (24 \times 14) + 0.5 \times (18 \times 9) + 0.25 \times (2 \times 5) - (15.5 \times 9.25) = 167.5 - 143.375 = 24.125.
\]

Correlation:

\[
\rho[R_A, R_B] = \frac{\sigma[R_A, R_B]}{\sigma[R_A] \sigma[R_B]} = \frac{24.125}{8.1701 \times 3.1918} = 0.9251
\]

2. asset A and the riskless asset?
Covariance:
\[ \sigma[R_A, R_f] = 0.25 \times (24 \times 7) + 0.5 \times (18 \times 7) + 0.25 \times (2 \times 7) - (15.5 \times 7) = 108.5 - 108.5 = 0. \]
Correlation:
\[ \rho[R_A, R_f] = \frac{\sigma[R_A, R_f]}{\sigma[R_A] \sigma[R_f]} = 0/0 \text{ which is not well defined.} \]

3. asset A and the riskless asset?

Covariance:
\[ \sigma[R_B, R_f] = 0. \]
Correlation:
\[ \rho[R_B, R_f] = \frac{\sigma[R_B, R_f]}{\sigma[R_B] \sigma[R_f]} = 0/0 \text{ which is not well defined.} \]

D. What is the expected return and standard deviation of return of a portfolio consisting of \( \omega\% \) invested in asset A and \((1-\omega)\% \) in the riskless asset when \( \omega \% \) is
1. -20%?
2. 60%?
3. 120%?

As an illustration, for \( \omega_{A,p} = -0.2 \):
\[
E[R_p] = \omega_{A,p} E[R_A] + (1 - \omega_{A,p}) R_f \\
= -0.2 \times 15.5\% + 1.2 \times 7\% = 5.3\%
\]
and
\[
\sigma[R_p] = |\omega_{A,p}| \sigma[R_A] \\
= |-0.2| \times 8.1701\% = 1.6340\%
\]

<table>
<thead>
<tr>
<th>( \omega_{A,p} )</th>
<th>( E[R_p] )</th>
<th>( \sigma[R_p] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>5.3%</td>
<td>1.6340%</td>
</tr>
<tr>
<td>0.6</td>
<td>12.1%</td>
<td>4.9020%</td>
</tr>
<tr>
<td>1.2</td>
<td>17.2%</td>
<td>9.8041%</td>
</tr>
</tbody>
</table>

E. What is the expected return and standard deviation of return of a portfolio consisting of \( \omega\% \) invested in asset B and \((1-\omega)\% \) in the riskless asset when \( \omega \% \) is
1. -20%?
2. 60%?
3. 120%?

As an illustration, for \( \omega_{B,p} = 1.2 \):
\[
E[R_p] = \omega_{B,p} E[R_B] + (1 - \omega_{B,p}) R_f \\
= 1.2 \times 9.25\% + -0.2 \times 7\% = 9.7\%
\]
and
\[ \sigma[R_p] = |\omega_{B,p}| \sigma[R_B] \\
= |1.2| \times 3.1918\% = 3.8301\% \]

<table>
<thead>
<tr>
<th>\omega_{B,p}</th>
<th>E[R_p]</th>
<th>\sigma[R_p]</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>6.55%</td>
<td>0.6383%</td>
</tr>
<tr>
<td>0.6</td>
<td>8.35%</td>
<td>1.9151%</td>
</tr>
<tr>
<td>1.2</td>
<td>9.7%</td>
<td>3.8301%</td>
</tr>
</tbody>
</table>

F. If a risk-averse investor has to decide whether to hold either asset A with the riskless asset or asset B with the riskless asset, which asset would the investor prefer to hold in combination with the riskless asset? Explain why? Do you need more information about the investor’s preferences to answer the question?

Any risk-averse individual prefers the risky asset whose CAL has the higher slope. The reason is that for any point on the lower-sloped CAL, there exists a point on the higher-sloped CAL with the same expected return but lower standard deviation.

slope-CAL(A) = \frac{E[R_A] - R_f}{\sigma[R_A]} = \frac{15.5\% - 7\%}{8.1701\%} = 1.0404.

slope-CAL(B) = \frac{E[R_B] - R_f}{\sigma[R_B]} = \frac{9.25\% - 7\%}{3.1918\%} = 0.7049.

So any risk-averse individual prefers to hold asset A in combination with the riskless asset than asset B.

G. What is the expected return and standard deviation of return of a portfolio consisting of \( \omega \)\% invested in asset A and \((1-\omega)\% in asset B when \( \omega \) is

1. -20%?
2. 80%?
3. 120%?

As an illustration, for \( \omega_{A,p} = 0.8 \):
\[
E[R_p] = \omega_{A,p} E[R_A] + (1- \omega_{A,p} ) E[R_B] = 0.8 \times 15.5\% + 0.2 \times 9.25\% = 14.25\%
\]
and
\[
\sigma[R_p]^2 = \omega_{A,p}^2 \sigma[R_A]^2 + \omega_{B,p}^2 \sigma[R_B]^2 + 2 \omega_{A,p} \omega_{B,p} \sigma[R_A, R_B] = (0.8x0.8) 66.75 + (0.2x0.2) 10.1875 + 2 (0.8x0.2) 24.125 = 42.72 + 0.4075 + 7.72 = 50.8475
\]
\[
\sigma[R_p] = 7.1307\%.
\]
<table>
<thead>
<tr>
<th>$\omega_{A,p}$</th>
<th>$E[R_p]$</th>
<th>$\sigma[R_p]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>8%</td>
<td>2.4%</td>
</tr>
<tr>
<td>0.8</td>
<td>14.25%</td>
<td>7.1307%</td>
</tr>
<tr>
<td>1.2</td>
<td>16.75%</td>
<td>9.2167%</td>
</tr>
</tbody>
</table>

Problem IV: Asset A is given by $\times$ and Asset B is given by $+$.
II. *Using Dividend Yield Information*: Suppose the following data is to be used by Ms Q (a risk-averse investor) to form a portfolio that consists of the small firm fund and T-bills.

\[
\begin{align*}
E[R_{\text{Small}}(t)] &= 1.369 \\
\sigma[R_{\text{Small}}(t)] &= 8.779 \\
E[DP(\text{start } t)] &= 4.446 \\
\sigma[DP(\text{start } t)] &= 1.513 \\
\sigma[DP(\text{start } t), R_{\text{Small}}(t)] &= 1.967
\end{align*}
\]

where \( DP(\text{start } t) \) is the dividend yield on the S&P 500 known at the start of month \( t \). \( R_{\text{Small}}(t) \) is the return on the small firm fund in month \( t \).

A. What is the intercept and slope coefficients from a regression of \( R_{\text{Small}}(t) \) (dependent variable) on \( DP(\text{start } t) \)?

Slope: \( \varphi_{\text{Small}, DP} = \frac{\sigma[R_{\text{Small}}(t), DP(\text{start } t)]}{\sigma[DP(\text{start } t)]^2} = \frac{1.967}{(1.513 \times 1.513)} = 0.859 \)

Intercept: \( \mu_{\text{Small}, DP} = E[R_{\text{Small}}(t)] - \varphi_{\text{Small}, DP} E[DP(\text{start } t)] = 1.369 - 0.859 \times 4.446 = -2.451 \).

Note that the standard deviation of the residual from the regression of \( R_{\text{Small}}(t) \) on \( DP(\text{start } t) \) is 8.682.

B. Suppose it is the end of March 1997, Ms Q does not know \( DP \) and the return on T-bills for April is 0.3%.

1. What is the expected April return on the small firm fund?

\[ E[R_{\text{Small}}(t)] = 1.369\% \]

2. Will Ms Q short sell the small firm fund?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. \( E[R_{\text{Small}}(t)] > R_f \). So Ms Q does not want to short sell.

3. Will Ms Q buy the small firm fund on margin?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. \( E[R_{\text{Small}}(t)] > R_f \). So Ms Q may want to buy on margin depending on how risk averse she is.

4. Will Ms Q buy a positive amount of both assets?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. \( E[R_{\text{Small}}(t)] > R_f \). So Ms Q may want to buy positive amounts of both depending on how risk
C. Suppose it is the end of March 1997, Ms Q knows that DP is 2 and the return on T-bills for April is 0.3%.

1. What is the expected April return on the small firm fund?

$$\mu_{\text{Small,DP}} + \varphi_{\text{Small,DP}} \text{DP(start Apr)} = -2.451 + 0.859 \times 2 = -0.733\%.$$ 

2. Will Ms Q short sell the small firm fund?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. $E[R_{\text{Small}(t)}] < R_F$. So Ms Q does want to short sell.

3. Will Ms Q buy the small firm fund on margin?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. $E[R_{\text{Small}(t)}] < R_F$. So Ms Q does not want to buy on margin.

4. Will Ms Q buy a positive amount of both assets?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. $E[R_{\text{Small}(t)}] < R_F$. So Ms Q does not want to buy positive amounts of both.

D. Suppose it is the end of October 1997, Ms Q does not know DP and the return on T-bills for November is 0.4%.

1. What is the expected November return on the small firm fund?

2. Will Ms Q short sell the small firm fund?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. $E[R_{\text{Small}(t)}] > R_F$. So Ms Q does not want to short sell.

The answer to this question is the same as for part B.

E. Suppose it is the end of October 1997, Ms Q knows that DP is 5 and the return on T-bills for November is 0.4%.

1. What is the expected November return on the small firm fund?

$$\mu_{\text{Small,DP}} + \varphi_{\text{Small,DP}} \text{DP(start Nov)} = -2.451 + 0.859 \times 5 = 1.844\%.$$ 

2. Will Ms Q short sell the small firm fund?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. $E[R_{\text{Small}(t)}] > R_F$. So Ms Q does not want to short sell.
3. Will Ms Q buy the small firm fund on margin?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. 
E[R\text{Small}(t)]>R_f. So Ms Q may want to buy on margin depending on how risk averse she is.

4. Will Ms Q buy a positive amount of both assets?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. 
E[R\text{Small}(t)]>R_f. So Ms Q may want to buy positive amounts of both depending on how risk averse she is.

III. The Two Risky Asset Case:

A. As an illustration, for \( \omega_{S,p} = 0.6 \):

\[
E[R_p] = \omega_{S,p} E[R_S] + (1 - \omega_{S,p}) E[R_B] = 0.6 \times 22\% + 0.4 \times 13\% = 18.4\% 
\]

and

\[
\sigma[R_p]^2 = \omega_{S,p}^2 \sigma[R_S]^2 + \omega_{B,p}^2 \sigma[R_B]^2 + 2 \omega_{S,p} \omega_{B,p} \rho[R_S, R_B] \sigma[R_S] \sigma[R_B] 
\]

\[
= (0.6 \times 0.6) (32 \times 32) + (0.4 \times 0.4) (23 \times 23) + 2 (0.6 \times 0.4) (0.15 \times 32 \times 23) 
\]

\[
= 368.64 + 84.64 + 52.992 = 506.272 
\]

\[
\sigma[R_p] = 22.5005\%. 
\]

<table>
<thead>
<tr>
<th>( \omega_{A,p} )</th>
<th>( E[R_p] )</th>
<th>( \sigma[R_p] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>13%</td>
<td>23%</td>
</tr>
<tr>
<td>0.2</td>
<td>14.8%</td>
<td>20.3678%</td>
</tr>
<tr>
<td>0.4</td>
<td>16.6%</td>
<td>20.1810%</td>
</tr>
<tr>
<td>0.6</td>
<td>18.4%</td>
<td>22.5005%</td>
</tr>
<tr>
<td>0.8</td>
<td>20.2%</td>
<td>26.6805%</td>
</tr>
<tr>
<td>1.0</td>
<td>22%</td>
<td>32%</td>
</tr>
</tbody>
</table>

B. Note that ! denotes the tangency portfolio T in the following graph.
C.

Use the following formula:

$$\omega_{S,T} = \frac{\sigma[R_B]^2 E[r_S] - \sigma[R_S, R_B] E[r_B]}{\{\sigma[R_B]^2 E[r_S] - \sigma[R_S, R_B] E[r_B]\} + \{\sigma[R_S]^2 E[r_B] - \sigma[R_S, R_B] E[r_S]\}}$$

where \(r_i = R_i - R_f\) is the excess return on asset \(i\) (in excess of the riskless rate).

Now

\(E[r_S] = 22\% - 9\% = 13\%\).

\(E[r_B] = 13\% - 9\% = 4\%\).

\(\sigma[R_S]^2 = 32 \times 32 = 1024\).
\( \sigma[R_b]^2 = 23 \times 23 = 529. \)
\( \sigma[R_s, R_b] = 0.15 \times 32 \times 23 = 110.4. \)

So, the weight of S in the tangency portfolio T is given by
\[
\omega_{S,T} = \frac{\{529 \times 13 - 110.4 \times 4\}}{\{529 \times 13 - 110.4 \times 4\} + \{1024 \times 4 - 110.4 \times 13\}}
\]
\[
= \frac{6435.4}{6435.4 + 2660.8} = 0.7075,
\]
and the weight of B in the tangency portfolio T is
\[
\omega_{B,T} = (1 - \omega_{S,T}) = 0.2925.
\]

Thus,
\[
E[R_T] = \omega_{S,T} E[R_S] + (1 - \omega_{S,T}) E[R_B]
\]
\[
= 0.7075 \times 22\% + 0.2925 \times 13\% = 19.3671\%
\]
and
\[
\sigma[R_T]^2 = \omega_{S,T}^2 \sigma[R_S]^2 + \omega_{B,T}^2 \sigma[R_B]^2 + 2 \omega_{S,T} \omega_{B,T} \sigma[R_S, R_B]
\]
\[
= (0.7075^2 \times 0.7075) 1024 + (0.2925^2 \times 0.2925) 529 + 2 (0.7075 \times 0.2925) 110.4
\]
\[
= 512.57 + 45.26 + 45.69 = 603.52
\]
\[
\sigma[R_T] = 24.5667\%.
\]

D. The reward to variability ratio is just the slope of the CAL.
\[
slope-CAL(T) = \frac{E[R_T] - R_f}{\sigma[R_T]} = \frac{19.3671\%-9\%}{24.5667\%} = 0.4220.
\]

E.

1. a. One Answer: Any portfolio on the CAL of the risky tangency portfolio T consists of T and the riskless asset. So for fund p,
\[
E[R_p] = 15\% = \omega_{T,p} E[R_T] + (1 - \omega_{T,p}) R_f
\]
\[
= R_f + \omega_{T,p} \{E[R_T] - R_f\}
\]
\[
= 9\% + \omega_{T,p} \{19.3671\%-9\%\}
\]
which implies that the weight of the tangency portfolio T in fund p is
\[
\omega_{T,p} = 6\%/10.3671\% = 0.5788.
\]

Finally,
\[
\sigma[R_p] = |\omega_{T,p}| \sigma[R_T]
\]
\[
= 0.5788 \times 24.5667\% = 14.22\%.
\]

b. Second Answer: The equation for the CAL (T) line is given by
\[
E[R_p] = R_f + \sigma(R_p) \{E[R_T] - R_f\}/ \sigma[R_T].
\]
Thus,
\[
15\% = 9\% + \sigma(R_p) \{19.3671\%-9\%\}/24.5667\%
\]
which implies
\[
\sigma[R_p] = 6\%/0.4220 = 14.22\%.
\]
2. Know that
\[ R_p = \omega_{T,p} R_T + (1 - \omega_{T,p}) R_f \]
where \( \omega_{T,p} = 0.5788 \) is the weight of the tangency portfolio \( T \) in the fund \( p \) and
\[ R_T = \omega_{S,T} R_S + (1 - \omega_{S,T}) R_B \]
where \( \omega_{S,T} = 0.7075 \) is the weight of the stock fund \( S \) in the tangency portfolio \( T \).

So
\[ R_p = \omega_{T,p} \{ \omega_{S,T} R_S + (1 - \omega_{S,T}) R_B \} + (1 - \omega_{T,p}) R_f \]
giving
\[ \omega_{S,p} = \omega_{T,p} \omega_{S,T} = 0.5788 \times 0.7075 = 0.4095 \] as the weight of the stock fund \( S \) in fund \( p \).
\[ \omega_{B,p} = \omega_{T,p} (1 - \omega_{S,T}) = 0.5788 \times 0.2925 = 0.1693 \] as the weight of the bond fund \( B \) in fund \( p \).
\[ \omega_{f,p} = (1 - \omega_{T,p}) = 0.4212 \] as the weight of the riskless asset in fund \( p \).

F. If the fund \( q \) consists only of the stock fund and the bond fund:
\[ E[R_q] = 15\% = \omega_{S,q} E[R_S] + (1 - \omega_{S,q}) E[R_B] \]
\[ = E[R_S] + \omega_{S,q} \{ E[R_S] - E[R_B] \} \]
\[ = 13\% + \omega_{S,q} 9\% \]
which implies that the weight of the stock fund \( S \) in the fund \( q \) is given by
\[ \omega_{S,q} = \frac{2\%}{9\%} = 0.2222 \]
and the weight of the bond fund \( B \) in \( q \) is
\[ \omega_{B,q} = (1 - \omega_{S,q}) = 0.7778. \]

So fund \( q \)'s standard deviation is
\[ \sigma[R_q]^2 = \omega_{S,q}^2 \sigma[R_S]^2 + \omega_{B,q}^2 \sigma[R_B]^2 + 2 \omega_{S,q} \omega_{B,q} \sigma[R_S, R_B] \]
\[ = (0.2222 \times 0.2222) (1024) + (0.7778 \times 0.7778) (529) + 2 (0.2222 \times 0.7778) (110.4) \]
\[ = 50.558 + 320.031 + 38.160 = 408.749 \]
\[ \sigma[R_q] = 20.2175\%. \]

Can see that though fund \( p \) from the previous question and fund \( q \) both have the same expected return, fund \( p \) has the lower standard deviation. So any risk averse individual would prefer to hold fund \( p \).
IV. **SML and the CAPM:**

A. In a CAPM world, all assets lie on the SML. So

\[ E[R_p] = R_f + \beta_{p,m} \{E[R_m] - R_f \} \]

\[ 20\% = 5\% + \beta_{p,m} \{15\% - 5\% \} \]

\[ \beta_{p,m} = \frac{15\%}{10\%} = 1.5. \]

B. 

1. The market has a Beta with respect to the market of 1. All assets plot on the SML including the market. So the market has the same expected return as the portfolio with a \( \beta_{p,m} \) of 1.

2. All assets lie on the SML. So an asset with a \( \beta_{p,m} = 0 \) has an expected return of:

\[ E[R_p] = R_f + \beta_{p,m} \{E[R_m] - R_f \} = 4\% + 0 \{12\% - 4\% \} = 4\%. \]

3. The expected return on the stock is given by:

\[ E[R] = R_f + \beta \{E[R_m] - R_f \} = 4\% + 0.5 \{12\% - 4\% \} = 0\%. \]

The intrinsic value of the stock is given by:

\[ V_0 = \frac{E[P_1 + D_1]}{(1 + E[R])} = \frac{41 + 3}{1.00} = 44 \]

which is greater than its current price. So it is underpriced today.

V. **SML vs CML in the CAPM:** Assume that the CAPM holds in the economy. The following data is available about the market portfolio, the riskless rate and two assets, A and B. Remember \( \beta_{i,m} = \sigma[R_i, R_m]/(\sigma[R_m]^2) \).

<table>
<thead>
<tr>
<th>Asset i</th>
<th>( E[R_i] )</th>
<th>( \sigma[R_i] )</th>
<th>( \beta_{i,m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>m (market)</td>
<td>0.15</td>
<td>0.08</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>0.096</td>
<td>1.2</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>0.07</td>
<td>0.6</td>
<td></td>
</tr>
</tbody>
</table>

\( R_f = 0.10. \)

A. What is \( \beta_{i,m} \) for i equal to the market portfolio (i.e., \( \beta_{m,m} \))? 

\[ \beta_{m,m} = \frac{\sigma[R_m, R_m]}{(\sigma[R_m]^2)} = 1. \]

B. What is the expected return on asset A (i.e., \( E[R_A] \))?
All assets plot on the SML:
\[ E[R_i] = R_f + \beta_{i,M} \{E[R_M] - R_f \} \]
So
\[ E[R_A] = R_f + \beta_{A,M} \{E[R_M] - R_f \} = 0.10 + 1.2 \{0.15-0.10\} = 0.16. \]

C. What is the expected return on asset B (i.e., \( E[R_B] \))?

Similarly,
\[ E[R_B] = R_f + \beta_{B,M} \{E[R_M] - R_f \} = 0.10 + 0.6 \{0.15-0.10\} = 0.13. \]

D. Does asset A plot:
1. on the SML (security market line)?
   Yes.
2. on the CML (capital market line)?
   Yes.

Formula for the CML:
\[ E[R_i] = R_f + \sigma[R_i] \{E[R_M] - R_f \} / \sigma[R_M]. \]
For A,
\[ R_f + \sigma[R_A] \{E[R_M] - R_f \} / \sigma[R_M] = 0.10 + 0.096 \{0.15-0.10\} / 0.08 = 0.16 = E[R_A] \]
as required for A to lie on the CML.

E. Does asset B plot:
1. on the SML?
   Yes.
2. on the CML?
   No since it does not lie on CML.

For B,
\[ R_f + \sigma[R_B] \{E[R_M] - R_f \} / \sigma[R_M] = 0.10 + 0.07 \{0.15-0.10\} / 0.08 = 0.14375 > 0.13 = E[R_B] \]
and so B does not lie on CML.

F. Could any investor be holding asset A as her entire portfolio?
   Yes since it lies on the CML.

G. Could any investor be holding asset B as her entire portfolio?
   No since it does not lie on the CML.

H. What is the correlation of asset A with the market portfolio?
Recall \( \beta_{i,M} = \rho[R_i, R_M] \sigma[R_i] / \sigma[R_M] \) which implies \( \rho[R_i, R_M] = \beta_{i,M} \sigma[R_M] / \sigma[R_i]. \)
So, for A,
\[ \rho[R_A, R_M] = \beta_{A,M} \frac{\sigma[R_M]}{\sigma[R_A]} = \frac{(1.2 \times 0.08)}{0.096} = 1. \]

I. What is the correlation of asset B with the market portfolio?

Similarly, for B,
\[ \rho[R_B, R_M] = \beta_{B,M} \frac{\sigma[R_M]}{\sigma[R_B]} = \frac{(0.6 \times 0.08)}{0.07} = 0.6857. \]

J. Can anything be said about the composition of asset A (i.e., what assets make up asset A)?

Since A lies on the CML, it must be some combination of the market portfolio and the riskless asset.

K. Can anything be said about the composition of asset B?

No.
Problem Set Solution: ICAPM, Market Efficiency and Valuation.

I. ICAPM. Let \( \text{TERM(Jan)} \) be the difference in the yield on a long term hi-grade corporate bond and 1 month T-bill at the end of January. Suppose each individual cares about \( \{E[R_p(Jan)], \sigma[R_p(Jan)], \sigma[R_p(Jan), \text{TERM(Jan)}]\} \) when forming his/her portfolio \( p \) for January. The following additional information is available:

<table>
<thead>
<tr>
<th>i</th>
<th>( E[R_i(Jan)] )</th>
<th>( \beta^*_{i,M} )</th>
<th>( \beta^*_{i,\text{TERM}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>18%</td>
<td>1.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Yellow</td>
<td>1.1</td>
<td>1.2</td>
<td></td>
</tr>
</tbody>
</table>

where \( \beta^*_{i,M} \) and \( \beta^*_{i,\text{TERM}} \) are regression coefficients from a regression of \( R_i(Jan) \) on \( R_M(Jan) \) and \( \text{TERM(Jan)} \):

\[
R_i(Jan) = \varphi_{i,0} + \beta^*_{i,M} R_M(Jan) + \beta^*_{i,\text{TERM}} \text{TERM(Jan)} + e_i
\]

Also know that \( E[R_M(Jan)] = 14\% \) and \( R_f(Jan) = 8\% \).

A. What is the expected January return for Yellow?

Since each individual cares about \( \{E[R_p(Jan)], \sigma[R_p(Jan)], \sigma[R_p(Jan), \text{TERM(Jan)}]\} \) when forming his/her portfolio \( p \) for January, it follows that any asset satisfies:

\[
E[R_i(Jan)] = R_f(Jan) + \beta^*_{i,M} \lambda^*_M + \beta^*_{i,\text{TERM}} \lambda^*_\text{TERM}
\]

Also know that \( \lambda^*_M = E[R_M(Jan)]-R_f. \)

So

\[
E[R_{\text{Red}(Jan)}] = R_f(Jan) + \beta^*_{\text{Red,M}} \lambda^*_M + \beta^*_{\text{Red,TERM}} \lambda^*_\text{TERM}
\]

\[
18\% = 8\% + 1.4 \times (14\% - 8\%) + 0.4 \times \lambda^*_\text{TERM}
\]

which implies

\(
\lambda^*_\text{TERM} = \frac{1.6\%}{0.4} = 4\%.
\)

So then the expected return on Yellow can be calculated

\[
E[R_{\text{Yellow}(Jan)}] = R_f(Jan) + \beta^*_{\text{Yellow,M}} \lambda^*_M + \beta^*_{\text{Yellow,TERM}} \lambda^*_\text{TERM}
\]

\[
= 8\% + 1.1 \times 6\% + 1.2 \times 4\%
\]

\[
= 19.4\%.
\]

B. What is the risk premium for bearing TERM risk?

As calculated above, \( \lambda^*_\text{TERM} = 4\%. \)
C. Is the market on the minimum variance frontier? Why or why not?

No. Since all assets satisfy
\[ E[R_i(Jan)] = R_f(Jan) + \beta_{i,M} \lambda_M + \beta_{i,TERM} \lambda_{TERM}, \]
it follows that
\[ E[R_i(Jan)] = \frac{E[R_{0,M}(Jan)] + \beta_{i,M} \{E[R_{M}(Jan)] - E[R_{0,M}(Jan)]\}}{E[R_{0,M}(Jan)] + \beta_{i,M} \{E[R_{M}(Jan)] - E[R_{0,M}(Jan)]\}}, \]
and so using minimum variance mathematics, M is not on the MVF.

D. Give one reason why an individual may care about the covariance of her portfolio return with TERM(Jan).

TERM(Jan) may be correlated with expected February asset returns (at the start of February).

II. Market Efficiency. Answer.

A. No. The price may fully reflect all information contained in past prices and be weak form efficient even though price does not reflect all information in IBM’s annual report.

B. Yes. Since IBM’s annual report is publicly available after its release date, price does not fully reflect all publicly available information and so the market is not semi strong form efficient.

C. Yes. If the price does not reflect all publicly available information then it does not reflect all existing information. Thus, the market is not strong form efficient.

III. Equity Valuation, the Dividend Discount Model and ROE.

A.
ROE = 16%; b = 0.5; E[E_i] = 2; E[R] = 12%.

1. So
\[ P_0 = \frac{E[D_1]}{E[R] - g} = \frac{E[E_i](1 - b)}{E[R] - b \times ROE} = \frac{2 \times (1 - 0.5)}{0.12 - 0.5 \times 0.16} = 25. \]

2.
\[ g = b \times ROE = 0.5 \times 0.16 = 0.08. \]
\[ E[D_1] = E[E_i](1 - b) = 2 \times (1 - 0.5) = 1 \]
\[ E[D_2] = E[D_1](1 + g)^3 = 1 \times 1.08^3 = 1.2597. \]
\[ E[P_3] = \frac{E[D_4]}{E[R] - g} = \frac{1.2597}{0.12 - 0.08} = 31.49. \]

B. Constant growth DDM model says
\[ P_0 = \frac{E[D_1]}{E[R] - g} \]
which implies
\[ E[R] = \frac{E[D_1]}{P_0} + g = 0.60/20 + 0.08 = 0.11. \]
C.

1. 
\[ P_0 = \frac{E[D_1]}{E[R] - g} = \frac{8}{0.1 - 0.05} = 160. \]

2. 
Know 
\[ 1 - b = \frac{E[D_1]}{E[E_1]}; \]
and so 
\[ b = 1 - \frac{E[D_1]}{E[E_1]} = 1 - \frac{8}{12} = 1/3. \]
Also know 
\[ g = ROE \times b \]
and so 
\[ ROE = \frac{0.05}{1/3} = 0.15 \text{ or } 15\%. \]

3. 
If the firm set \( b = 0 \) and paid out all future earnings as dividends, share price would be 
\[ E[E_1]/E[R] = \frac{12}{0.1} = 120. \]
Thus, the present value of growth opportunities is given by 
\[ 160 - 120 = 40. \]

IV.  
**Equity Valuation, Asset Composition and Leverage.**

A. 
Know 
\[ \beta_{SIBX} = \{V_{IBX}/S_{IBX}\} \beta_{VIBX} = \{6/(6-1)\} 1.3 = 1.56. \]

Using the SML, 
\[ E[R_{SIBX}] = R_F + \beta_{SIBX} (E[R_M] - R_F) = 10\% + 1.56 \times (18\% - 10\%) = 22.48\% \]

Use the constant growth DDM 
\[ p_{SIBX,ex} = D_{SIBX,ex} (1+g_{SIBX}) /\{E[R_{SIBX}] - g_{SIBX}\} = 3.76 \times (1.0648)/(0.2248 - 0.0648) = 25. \]

B. 
\[ n_{IBX} = S_{IBX} / p_{SIBX} = 5M/25 = 0.2 \text{ M shares.} \]
Problem Set 4 Solution: Fixed Income Valuation.

I.  

*Implied Yield Curve, Forward Rates and No Arbitrage:* Consider the following prices for U.S. treasury notes on 2/15/96.

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>4½</td>
<td>Aug 96</td>
<td>98:11</td>
</tr>
<tr>
<td>5¼</td>
<td>Feb 97</td>
<td>99:01</td>
</tr>
<tr>
<td>5¾</td>
<td>Aug 97</td>
<td>98:23</td>
</tr>
<tr>
<td>6</td>
<td>Feb 98</td>
<td>98:15</td>
</tr>
</tbody>
</table>

A. What is the implied yield curve (expressed in terms of APRs with semiannual compounding)?

The following formula relating the yield on a discount bond to the relevant discount factor is used throughout this question. Let \( d_t(0) \) be the discount factor for a \( t \)-year discount bond, and \( y_t(0) \) be the yield on a \( t \) year discount bond expressed as an APR with semiannual compounding:

\[
y_t(0) = 2 \left\{ \frac{1}{d_t(0)} \right\}^{\frac{1}{2t}} - 1 \quad \Leftrightarrow \quad d_t(0) = \frac{1}{\left[1 + \frac{y_t(0)}{2}\right]^{2t}}.
\]

To obtain the yield on a 6 month discount bond:

a. Can recover the discount factor on a 6-month discount bond using the Aug 96 note since it has only one payment left on 8/15/96:

\[
d_{\frac{1}{2}} \text{ (Feb 96)} = \frac{98.3438}{100 + [4\frac{1}{2}/2]} = 0.96180
\]

b. Can convert the 6-month discount bond discount factor into a yield expressed as an APR with semi-annual compounding:

\[
y_{\frac{1}{2}} \text{ (Feb 96)} = \{[1/d_{\frac{1}{2}} \text{ (Feb 96)}] - 1\} \times 2 = \{[1/0.96180] - 1\} \times 2 = 7.9440\%.
\]

Can recover the yield on a 1 year discount bond by creating a synthetic 1 year discount bond:

1. More specifically:
   1. let \( b \) be the number of 5¼ Feb 97 notes bought.
   2. let \( a \) be the number of 4½ Aug 96 notes bought.
Problem Set Solutions  
Foundations of Finance

<table>
<thead>
<tr>
<th>Position</th>
<th>2/15/96</th>
<th>8/15/96</th>
<th>2/15/97</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy $a$ 4½ Aug 96 notes</td>
<td>-$a \times 98.34375$</td>
<td>$a \times 102.25$</td>
<td></td>
</tr>
<tr>
<td>Buy $b$ 5¼ Feb 97 notes</td>
<td>$-b \times 99.03125$</td>
<td>$b \times 2.625$</td>
<td>$b \times 102.625$</td>
</tr>
<tr>
<td>Net</td>
<td>$-b \times 99.03125$</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>$-a \times 98.34375$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(2) So $b \times 102.625 = 100$ implies $b = 0.97442$.
(3) So $b \times 2.625 + a \times 102.25 = 0$ implies $a = -0.02501$.
(4) Thus, the cost of the synthetic discount bond is $0.97442 \times 99.03125 - 0.02501 \times 98.34375 = 94.038$.

c. The discount factor and yield on a 1 year discount bond can then be obtained (using the above stated formulas):

$d_1$ (Feb 96) = $94.038/100 = 0.94038$.
$y_1$ (Feb 96) = $([1/d_1$ (Feb 96)$]^{0.5} - 1) \times 2 = ([1/0.94038]^{0.5} - 1) \times 2 = 6.2425\%$.

Can recover the yield on a 1½ year discount bond by creating a synthetic 1½ year discount bond:

(1) More specifically:

(a) let $c$ be the number of 5¼ Aug 97 notes bought.
(b) let $b$ be the number of 5¼ Feb 97 notes bought.
(c) let $a$ be the number of 4½ Aug 96 notes bought.
<table>
<thead>
<tr>
<th>Position</th>
<th>2/15/96</th>
<th>8/15/96</th>
<th>2/15/97</th>
<th>8/15/97</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy a 4½ Aug 96 notes</td>
<td>-a x 98.34375</td>
<td>a 102.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy b 5¼ Feb 97 notes</td>
<td>-b x 99.03125</td>
<td>b 2.625</td>
<td>b 102.625</td>
<td></td>
</tr>
<tr>
<td>Buy c 5¾ Aug 97 notes</td>
<td>-c x 98.71875</td>
<td>c 2.875</td>
<td>c 2.875</td>
<td>c 102.875</td>
</tr>
<tr>
<td>Net</td>
<td>-c 98.71875</td>
<td>0</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>- b 99.03125</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>- a 98.34375</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(2) So c 102.875 = 100 implies c = 0.97205.
(3) So c 2.875 + b 102.625 = 0 implies b = -0.02723.
(4) So c 2.875 + b 2.625 + a 102.25 = 0 implies a = -0.02663.
(5) Thus, the cost of the synthetic discount bond is

\[
0.97205 \times 98.71875 - 0.02723 \times 99.03125 - 0.02663 \times 98.34375 = 90.644.
\]

The discount factor and yield for a 18 month discount bond can then be obtained:

d_{1.5} (Feb 96) = 90.644/100 = 0.90644; and,
y_{1.5} (Feb 96) = \left\{\frac{1}{d_{1.5} (Feb 96)} \right\}^{-1} \times 2 = \{[1/0.90644]^a -1\} \times 2 = 6.6571\%.

Can recover the yield on a 2 year discount bond by creating a synthetic 2 year discount bond:

(1) More specifically:
(a) let d be the number of 6 Feb 98 notes bought.
(b) let c be the number of 5¾ Aug 97 notes bought.
(c) let b be the number of 5¼ Feb 97 notes bought.
(d) let a be the number of 4½ Aug 96 notes bought.
Problem Set Solutions  Foundations of Finance

<table>
<thead>
<tr>
<th>Position</th>
<th>2/15/96</th>
<th>8/15/96</th>
<th>2/15/97</th>
<th>8/15/97</th>
<th>2/15/98</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy a</td>
<td>-a x 98.34375</td>
<td>a 102.25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4½ Aug 96 notes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy b</td>
<td>-b x 99.03125</td>
<td>b 2.625</td>
<td>b 102.625</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5¼ Feb 97 notes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy c</td>
<td>-c x 98.71875</td>
<td>c 2.875</td>
<td>c 2.875</td>
<td>c 102.875</td>
<td></td>
</tr>
<tr>
<td>5¼ Aug 97 notes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy d</td>
<td>-d 98.46875</td>
<td>d 3</td>
<td>d 3</td>
<td>d 3</td>
<td>d 103</td>
</tr>
<tr>
<td>6 Feb 98 notes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Net</td>
<td>-d 98.46875</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>-c 98.71875</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>- b 99.03125</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>- a 98.34375</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(2) So d 103 = 100 implies d = 0.97087
(3) So d 3 + c 102.875 = 0 implies c = -0.02831.
(4) So d 3 + c 102.875 + b 102.625 = 0 implies b = -0.02759.
(5) So d 3 + c 102.875 + b 2.625 + a 102.25 = 0 implies a = -0.02698.
(6) Thus, the cost of the synthetic discount bond is

\[
0.97087 \times 98.46875 - 0.02831 \times 98.71875 - 0.02759 \times 99.03125 - 0.02698 \times 98.34375 = 87.420.
\]

e. The discount factor and yield for a 2 year discount bond can then be obtained:

\[
d_2 \text{ (Feb 96)} = 87.420/100 = 0.87420; \text{ and,}
\]

\[
y_2 \text{ (Feb 96)} = \left\{\left[1/d_2 \text{ (Feb 96)}\right]^{1/2} -1\right\} \times 2 = \left\{[1/0.87420]^{1/2} -1\right\} \times 2 = 6.8364\%.
\]

B. What are the implied forward rates for the 6 month periods starting in 6 months, in 1 year and in 18 months (expressed as APRs with semiannual compounding)?

First, calculate \(d_{6/1}(\text{Feb 96})\), \(d_{1,1/2}(\text{Feb 96})\), and \(d_{1,1/2,2}(\text{Feb 96})\) using the formula

\[
d_{t,t+\tau}(0) = d_{t+\tau}(0) / d_t(0):
\]

a. \(d_{6/1}(\text{Feb 96}) = d_1(\text{Feb 96})/d_{6/1}(\text{Feb 96}) = 0.94038/0.96180 = 0.97773.

b. \(d_{1,1/2}(\text{Feb 96}) = d_{1/2}(\text{Feb 96})/d_1(\text{Feb 96}) = 0.90644/0.94038 = 0.96391.

c. \(d_{1/2,2}(\text{Feb 96}) = d_2(\text{Feb 96})/d_{1/2}(\text{Feb 96}) = 0.87420/0.90644 = \)
Then calculate the associated forward rates expressed as APRs with semiannual compounding using the following formula

\[ f_{t,t+\tau}(0) = 2 \left\{ \frac{1}{d_{t,t+\tau}(0)} \right\}^{1/(2\tau)} - 1 \] with \( \tau = \frac{1}{2} \).

a. \( f_{\frac{1}{2},1}(Feb 96) = 2 \left\{ \frac{1}{d_{\frac{1}{2},1}(Feb 96)} - 1 \right\} = 2 \left\{ \frac{1}{0.97773} - 1 \right\} = 4.5554\% \).

b. \( f_{1,1\frac{1}{2}}(Feb 96) = 2 \left\{ \frac{1}{d_{1,1\frac{1}{2}}(Feb 96)} - 1 \right\} = 2 \left\{ \frac{1}{0.96391} - 1 \right\} = 7.4883\% \).

c. \( f_{1\frac{1}{2},2}(Feb 96) = 2 \left\{ \frac{1}{d_{1\frac{1}{2},2}(Feb 96)} - 1 \right\} = 2 \left\{ \frac{1}{0.96443} - 1 \right\} = 7.3764\% \).

C. If there are no arbitrage opportunities, what is the price of a Aug 97 U.S. Treasury strip?

Use the discount factor for a 18 month discount bond:

\[ P_{Aug 97 \text{ strip}}(Feb 96) = d_{1\frac{1}{2}}(Feb 96) \times 100 = 0.906440 \times 100 = 90.6440. \]

D. Suppose the price of a Feb 97 U.S. Treasury strip is 94. Is there an arbitrage opportunity? If so, describe a strategy which earns an arbitrage profit.

Use the implied yield on a 1 year discount bond (expressed as an APR with semiannual compounding):

\[ P_{Feb 97 \text{ strip}}(Feb 96) = d_{1}(Feb 96) \times 100 = 0.940831 \times 100 = 94.038. \]

Since the price of the Feb 97 strip implied by the coupon bonds is greater than the strip’s actual price, you want to buy the Feb 97 strip and sell a synthetic Feb 97 strip created using the Aug 96 and Feb 97 coupon bonds.

Let \( a \) be the number of Feb 97 notes that you buy and \( b \) be the number of Aug 96 notes that you buy. Want to choose \( a \) and \( b \) so that the net cash flow at 2/15/97 is zero and the the net cash flow at 8/15/96 is zero:
<table>
<thead>
<tr>
<th>Position</th>
<th>2/15/96</th>
<th>8/15/96</th>
<th>2/15/97</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 1 Feb 97 strip</td>
<td>-94</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>Buy $a$ x Feb 97 note</td>
<td>-$a$ x 99.03125</td>
<td>$a$ x 2.625</td>
<td>$a$ x 102.625</td>
</tr>
<tr>
<td>Buy $b$ x Aug 96 note</td>
<td>-$b$ x 98.34374</td>
<td>$b$ x 102.25</td>
<td></td>
</tr>
<tr>
<td>Net</td>
<td>-94 - $a$ x 99.03125</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

So

a. $100 + a \times 102.625 = 0$ which implies $a = -100/102.625 = -0.97442$. Thus, the Feb 97 note is sold.

b. $a \times 2.625 + b \times 102.25 = 0$ which implies $b = -a \times 2.625/102.25 = 0.02502$.

Thus,

<table>
<thead>
<tr>
<th>Position</th>
<th>2/15/96</th>
<th>8/15/96</th>
<th>2/15/97</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 1 Feb 97 strip</td>
<td>-94</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>Sell 0.97442 Feb 97 notes</td>
<td>0.97442 x</td>
<td>-0.97442 x 2.625</td>
<td>-0.97442 x 102.625</td>
</tr>
<tr>
<td></td>
<td>99.03125</td>
<td>= -2.558</td>
<td>= -100</td>
</tr>
<tr>
<td></td>
<td>= 96.4980</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy 0.02502 Aug 96 notes</td>
<td>-0.02502 x</td>
<td>0.02502 x 102.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td>98.34374 = -2.4606</td>
<td>= 2.558</td>
<td></td>
</tr>
<tr>
<td>Net</td>
<td>0.03744</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

E. Suppose the prices for U.S. Treasury notes on 8/15/96 are given by:

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>5¼</td>
<td>Feb 97</td>
<td>98:11</td>
</tr>
<tr>
<td>5¾</td>
<td>Aug 97</td>
<td>98:21</td>
</tr>
<tr>
<td>6</td>
<td>Feb 98</td>
<td>98:00</td>
</tr>
</tbody>
</table>

1. What is the return from holding the Aug 97 note from 2/15/96 to 8/15/96?
The return from holding the Aug 97 note from 2/15/96 to 8/15/96 is given by:

\[
\frac{P_{5\% \text{ Aug 97}} (\text{Aug 96}) + C_{5\% \text{ Aug 97}} (\text{Aug 96}) - P_{5\% \text{ Aug 97}} (\text{Feb 96})}{P_{5\% \text{ Aug 97}} (\text{Feb 96})} = \frac{98.65625 + 2.875 - 98.71875}{98.71875} = 2.849\%.
\]

2. What is the return from holding the Aug 96 note from 2/15/96 to 8/15/96?

On 2/15/96, the Aug 96 note has an identical payoff to that of a 6 month discount bond. The return from holding the Aug 96 note from 2/15/96 to 8/15/96 is given by the yield on a 6 month discount bond on 2/15/96 expressed as an effective semiannual rate:

\[
y_{\text{s}}(\text{Feb 96})/2 = 7.9440%/2 = 3.9720\%.
\]

3. Calculate the implied yield curve (expressed in terms of APRs with semiannual compounding)?

Can recover the discount factor for and yield on a six month discount bond using the Feb 97 note since it has only one payment left on 2/15/97:

\[
d_{\text{s}} (\text{Aug 96}) = 98.34375/(100 + [5\%/2]) = 0.95828; \text{ and,}
\]

\[
y_{\text{s}} (\text{Aug 96}) = \{(1/d_{\text{s}} (\text{Aug 96})) -1\} x 2 = \{[1/0.95828] -1\} x 2 = 8.7067\%.
\]

Can recover the yield on a 1 year discount bond by creating a synthetic 1 year discount bond:

More specifically:

(a) let \(c\) be the number of 5\% Aug 97 notes bought.

(b) let \(b\) be the number of 5\% Feb 97 notes bought.

<table>
<thead>
<tr>
<th>Position</th>
<th>8/15/96</th>
<th>2/15/97</th>
<th>8/15/97</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy (b) 5% Feb 97 notes</td>
<td>BUY SYNTHETIC DISCOUNT BOND</td>
<td>(-b) 98.34375</td>
<td>(b) 102.625</td>
</tr>
<tr>
<td>Buy (c) 5% Aug 97 notes</td>
<td>-(c) 98.65625</td>
<td>(c) 2.875</td>
<td>(c) 102.875</td>
</tr>
<tr>
<td>Net</td>
<td>-(c) 98.65625</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>-(b) 98.34375</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(2) So \(c\) 102.875 = 100 implies \(c = 0.97205\).

(3) So \(c\) 2.875 + \(b\) 102.625 = 0 implies \(b = -0.02723\).

(4) Thus, the cost of the synthetic discount bond is

\[0.97205 \times 98.65625 - 0.02723 \times 98.34375 = 93.221.\]

(5) The discount factor for and yield on a 1 year discount bond can then be obtained:
Problem Set Solutions

\[ d_1 \text{ (Aug 96)} = 0.93221; \quad \text{and}, \]
\[ y_1 \text{ (Aug 96)} = \{\frac{1}{0.93221}\}^{0.5} - 1 \times 2 = 7.1443\% . \]

Can then recover the yield on a 1½ year discount bond by creating a synthetic 1½ year discount bond:

(1) More specifically:
(a) let \( d \) be the number of 6 Feb 98 notes bought.
(b) let \( c \) be the number of 5¼ Aug 97 notes bought.
(c) let \( b \) be the number of 5¼ Feb 97 notes bought.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Position} & \text{8/15/96} & \text{2/15/97} & \text{8/15/97} & \text{2/15/98} \\
\hline
\text{Buy } b & -b 98.34375 & b 102.625 \\
\text{5¼ Feb 97 notes} & & & & \\
\hline
\text{Buy } c & -c 98.65625 & c 2.875 & c 102.875 \\
\text{5¾ Aug 97 notes} & & & & \\
\hline
\text{Buy } d & -d 98 & d 3 & d 3 & d 103 \\
\text{6 Feb 98 notes} & & & & \\
\hline
\text{Net} & -d 98 & 0 & 0 & 100 \\
& -c 98.65625 & & & \\
& - b 98.34375 & & & \\
\hline
\end{array}
\]

(2) So \( d 103 = 100 \) implies \( d = 0.97087 \)
(3) So \( d 3 + c 102.875 = 0 \) implies \( c = -0.02831 \).
(4) So \( d 3 + c 2.875 + b 102.625 = 0 \) implies \( b = -0.02759 \).
(5) Thus, the cost of the synthetic discount bond is

\[
0.97087 \times 98 - 0.02831 \times 98.65625 - 0.02759 \times 98.34375 = 89.639.
\]

(6) The discount factor for and yield on a 1½ year discount bond can then be obtained:

\[ d_{1\frac{1}{2}} \text{ (Aug 96)} = 0.89639; \quad \text{and}, \]
\[ y_{1\frac{1}{2}} \text{ (Aug 96)} = \{\frac{1}{0.89639}\}^{0.5} - 1 \times 2 = 7.4263\% . \]

4. Consider 2/15/96's forward rate for the period 8/15/96 to 2/15/97. How does it compare to the 6 month interest rate on 8/15/96? If these two rates differ, discuss why.

\[ y_{6} \text{ (Aug 96)} = 8.7067\%; \quad \text{and}, \]
\[ f_{2,1}(\text{Feb 96}) = 4.554\%. \]
The two rates are different. But these rates need not be the same and generally will not be. Even if the expectations hypothesis holds and \( f^{*}_{t, t+\frac{1}{2}}(0) = \mathbb{E}_{t} y^{*}_{\frac{1}{2}}(t) \), it need not be the case that \( f^{*}_{t, t+\frac{1}{2}}(0) = y^{*}_{\frac{1}{2}}(t) \). The yield on a six month discount bond in Aug 96 depends on economic conditions at that time while the forward rate \( f^{*}_{t, t+\frac{1}{2}}(0) \) is set in Feb 96 and depends on expectations in Feb 96 about economic conditions in Aug 96.

II. \textit{Forward Rates and the Yield Curve.}

A. To determine the yield on a two year discount bond:

1. Use the Aug 95 strip to determine the \( \frac{1}{2} \)-year discount bond discount factor: 
   \[ d_{\frac{1}{2}}(2/15/95) = \frac{97.75}{100} = 0.9775. \]
2. Use the Feb 96 strip to determine the 1-year discount bond discount factor: 
   \[ d_{1}(2/15/95) = \frac{93.25}{100} = 0.9325. \]
3. Use the Aug 96 strip to determine the 1\( \frac{1}{2} \)-year discount bond discount factor: 
   \[ d_{\frac{3}{2}}(2/15/95) = \frac{90}{100} = 0.9. \]
4. Then use the 5\% Feb 97 government bond.
   a. The coupon of 2.5 paid in Aug 95 can be converted to a value today using 
      \[ d_{\frac{1}{2}}(2/15/95): \]
      \[ P_{\frac{1}{2}}(2/15/95) = 2.5 \times 0.9775 = 2.44375. \]
5. The coupon of 2.5 paid in Feb 96 can be converted to a value today using 
   \[ d_{1}(2/15/95): \]
   \[ P_{1}(2/15/95) = 2.5 \times 0.9325 = 2.33125. \]
6. The coupon of 2.5 paid in Aug 96 can be converted to a value today using 
   \[ d_{\frac{3}{2}}(2/15/95): \]
   \[ P_{\frac{3}{2}}(2/15/95) = 2.5 \times 0.9 = 2.25. \]
7. The value today of the final cash flow of 102.5 paid in Feb 97 can be obtained by subtracting the values of the earlier coupons from the bond’s price: 
   \[ P_{2}(2/15/95) = 98 - 2.44375 - 2.33125 - 2.25 = 90.975. \]
8. The yield on a two year discount bond expressed as an APR with semiannual compounding can be obtained: 
   \[ y_{2}(2/15/95) = \left\{\frac{102.5}{90.975}\right\} - 1 \times 2 = 0.06054 = 6.054\%. \]

B. Use the following formula to obtain the forward contract discount factor available today for the one year period starting in 6 months expressed as an EAR:

\[ d_{0.5,1.5}(2/15/95) = d_{1.5}(2/15/95)/d_{0.5}(2/15/95) = 0.9/0.9775 = 0.92072. \]

Then express this forward contract discount factor as an APR with semiannual compounding by using:

\[ f_{0.5,1.5}(2/15/95) = \left\{\frac{1}{d_{0.5,1.5}(2/15/95)}\right\}^{0.5} -1 \times 2 = \left\{\frac{1}{0.92072}\right\}^{0.5} -1 \times 2 = 8.433\%. \]
C. Now \( d_{0.5,1.5}(2/15/95) \) refers to the forward price at 2/15/95 for delivery of a 1 year-discount bond in 6 months. So \( d_{0.5,1.5}(2/15/95) \) refers to the forward price today of an Aug 96 strip to be delivered in 6 months time. The forward price at time 2/15/95 for a $100 face value strip can be calculated:

\[
d_{0.5,1.5}(2/15/195) \times 100 = 0.92072 \times 100 = 92.0717.
\]

III. Duration and Interest Rate Sensitivity:
A. Since the yield curve expressed as an APR with semiannual compounding is flat at 5.5%, \( y = 0.055 \). Thus,

\[
P_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} = \frac{(4\frac{3}{4}/2)}{1.0275} + \frac{(4\frac{3}{4}/2)}{(1.0275)^2} + \frac{(100 + 4\frac{3}{4}/2)}{(1.0275)^3}
\]

\[
= 2.3114 + 2.2496 + 94.3731 = 98.9341.
\]

B. Macaulay Duration:

\[
D_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} = \frac{0.5 \times (2.3114/98.9341) + 1 \times (2.2496/98.9341) + 1.5 \times (94.3731/98.9341)}{0.5 \times 0.023363 + 1 \times 0.022738 + 1.5 \times 0.953899} = 1.46527.
\]

“modified duration”:

\[
D_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} = \frac{D_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)}}{1+y/2} = \frac{1.46527/1.0275}{1.0275} = 1.42605.
\]

C. 1. Now, \( y^\Delta = 0.06 \). Thus, the new price is:

\[
P^\Delta_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} = \frac{(4\frac{3}{4}/2)}{1.03} + \frac{(4\frac{3}{4}/2)}{(1.03)^2} + \frac{(100 + 4\frac{3}{4}/2)}{(1.03)^3}
\]

\[
= 2.3058 + 2.2387 + 93.6876 = 98.2321.
\]

2. Now, \( \Delta y = 0.005 \). So

\[
\Delta P_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} \approx \Delta y \{ -P_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} \times D_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} / (1+y/2) \}
\]

\[
= 0.005 \{-98.9341 \times 1.46527 / 1.0275 \} = 0.005 \times -141.085 = -0.7054.
\]

Thus,

\[
P^\Delta_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} \approx P_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} + \Delta P_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} = 98.9341 - 0.7054 = 98.2287
\]

3. The price decline implied by duration is greater than the actual price decline.

D. 1. Now, \( y^\Delta = 0.05 \). Thus, the new price is:

\[
P^\Delta_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} = \frac{(4\frac{3}{4}/2)}{1.025} + \frac{(4\frac{3}{4}/2)}{(1.025)^2} + \frac{(100 + 4\frac{3}{4}/2)}{(1.025)^3}
\]

\[
= 2.3171 + 2.2606 + 95.0654 = 99.6430.
\]

2. Now, \( \Delta y = -0.005 \). So

\[
\Delta P_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} \approx \Delta y \{ -P_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} \times D_{4\frac{3}{4}} \text{ Feb 96 (Aug 94)} / (1+y/2) \}
\]

\[
= -0.005 \{-98.9341 \times 1.46527 / 1.0275 \} = -0.005 \times -141.085 = 0.7054.
\]
Thus, 
\[ P_{4\%\ Feb\ 96\ (Aug\ 94)} \approx P_{4\%\ Feb\ 96\ (Aug\ 94)} + \Delta P_{4\%\ Feb\ 96\ (Aug\ 94)} = 98.9341 + 0.7054 = 99.6395. \]

3. The price increase implied by duration is less than the actual price increase.

IV. *Immunization:*

A. Now, \( y = 0.06 \). Thus, 
\[
L_{(Aug\ 94)} = \frac{5M}{(1.03)^2} + \frac{5M}{(1.03)^4} + \frac{5M}{(1.03)^6} + \frac{5M}{(1.03)^8} + \frac{5M}{(1.03)^{10}} \\
\]

B. 

1. 
\[
D_L(Aug\ 94) = 1 \times \left( \frac{4.7130}{21.0103} \right) + 2 \times \left( \frac{4.4424}{21.0103} \right) \\
+ 3 \times \left( \frac{4.1874}{21.0103} \right) + 4 \times \left( \frac{3.9470}{21.0103} \right) + 5 \times \left( \frac{3.7205}{21.0103} \right) \\
= 2.8820.
\]

2. First need to determine the price of the note: 
\[
P_{6e\ Feb\ 96\ (Aug\ 94)} = \frac{(6e/2)/1.03 + (6e/2)/(1.03)^2 + (100+6e/2)/(1.03)^3}{3.2160 + 3.1223 + 94.5456} = 0.8839.
\]

Then can determine the note’s duration: 
\[
D_{6e\ Feb\ 96\ (Aug\ 94)} = \frac{0.5 \times (3.2160/100.8839) + 1 \times (3.1223/100.8839) + 1.5 \times (94.5456/100.8839)}{1.4527}.
\]

3. The duration of a discount bond is equal to its maturity. Thus, 
\[
D_{Aug\ 94\ strip\ (Aug\ 94)} = 10.
\]

C. To immunize the liability, need to match asset value to the liability value. So let \( \omega_{note} \) be the fraction of the $21.0103M invested in the note. Also need to match the duration of the assets to the duration of the liabilities: 
\[
D'_A\ (Aug\ 94) = D_L(Aug\ 94) = 2.8820.
\]

But 
\[
D'_A\ (Aug\ 94) = (1 - \omega_{note}) D_{Aug\ 94\ strip\ (Aug\ 94)} + \omega_{note} D_{6e\ Feb\ 96\ (Aug\ 94)}.
\]

So 
\[
2.8820 = (1 - \omega_{note}) 10 + \omega_{note} 1.4527
\]

which implies 
\[
\omega_{note} = \frac{(10 - 2.8820)}{(10 - 1.4527)} = 0.8328.
\]
So XYZ should invest $21.0103M \times 0.8328 = 17.4969M$ in the note and $3.5134M$ in the strip.
Problem Set Solutions

Problem Set 5 Solution: Derivatives

I. Options:
   A. BKM, Chapter 20, Question 6, part a.
   B. BKM, Chapter 20, Question 23.
   C. BKM, Chapter 20, Question 4.
   D. BKM, Chapter 20, Question 5.
   E. BKM, Chapter 20, Question 9, parts a, b, c and e.

II. Futures and Forward Contracts:
   A. BKM, Chapter 22, Question 6.
   B.

The spot price implied by the December futures price is

\[
\frac{360}{1+0.08} = 333.33
\]

which is greater than the spot price implied by the Jun futures price:

\[
\frac{346.3}{1+0.08}^{\frac{1}{2}} = 333.22
\]

So since the actual December futures price is too high, the arbitrage involves selling the December futures contract. An arbitrage position can be constructed as follows:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Today</th>
<th>Jun</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy gold today and sell in Dec</td>
<td>-S(0)</td>
<td></td>
<td>S(Dec)</td>
</tr>
<tr>
<td>Sell a Dec futures contract today</td>
<td>0</td>
<td></td>
<td>360 - S(Dec)</td>
</tr>
<tr>
<td>Sell Dec. T-bills with face value of 360 today and hold to maturity</td>
<td>(\frac{360}{1+0.08} = 333.33)</td>
<td>-360</td>
<td></td>
</tr>
<tr>
<td>Sell gold today and buy in Jun</td>
<td>S(0)</td>
<td></td>
<td>-S(Jun)</td>
</tr>
<tr>
<td>Buy a Jun futures contract today</td>
<td>0</td>
<td></td>
<td>S(Jun) - 346.3</td>
</tr>
<tr>
<td>Buy Jun. T-bills with face value of 346.3 today and hold to maturity</td>
<td>(-\frac{346.3}{1+0.08}^{\frac{1}{2}} = -333.22)</td>
<td>346.3</td>
<td></td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>0.11</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
C. Forward-spot parity implies a forward price that satisfies
\[ F_t(0) d_t(0) - C(t) d_x(0) = S(0) \]
\[ F_t(0)/1.05 + 1020*0.02 /1.05 = 1020 \]
\[ F_t(0) = 1050.6 \]
which is less than the available forward price.
Since the available forward price is too high, want to sell forward contracts and buy the underlying:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy index at 0 and sell at 1</td>
<td>-1020</td>
<td>S(1) + 1020*0.02</td>
</tr>
<tr>
<td>Sell a forward contract at 0 which delivers index at 1</td>
<td>0</td>
<td>1060 - S(1)</td>
</tr>
<tr>
<td>Sell 1-yr T-bills at 0 with face value of (1060+20.4)</td>
<td>1080.4/1.05</td>
<td>-1080.4</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>8.95</td>
<td>0</td>
</tr>
</tbody>
</table>

This strategy is an arbitrage opportunity.

D. BKM, Chapter 23, Question 5. For part c, assume that your portfolio consists of 60% in equities and 40% in T-bills.

E. The following formula converts discount bond discount factors to forward contract discount factors (it was introduced earlier in Lecture Note 15-17):
\[ d_{t,t+\tau}(0) = d_{t+\tau}(0) /d_t(0) \]
Is this formula consistent with the spot futures parity theorem? Explain why or why not. (Hint. Think about \( d_{t,t+\tau}(0) \) as referring to the forward price at time 0 for delivery of a \( \tau \)-discount bond at time \( t \).)

Consider spot forward parity for a forward contract to deliver at time \( t \) a discount bond maturing at \( t+\tau \) with a face value of $1. At time 0, the underlying is a \( (t+\tau) \) period discount bond so \( S(0) = d_{t+\tau}(0) \). There is no carrying costs or negative carring costs associated with holding a discount bond. So if \( F_t(0) \) is the forward price, spot forward parity says that:
\[ F_t(0) = [1+y*_{t+\tau}(0)]^\tau S(0). \]
So \( F_t(0) = [1+y*_{t+\tau}(0)]^\tau d_{t+\tau}(0) = [1/d_t(0)] d_{t+\tau}(0) \). Now, \( F_t(0) \) is the forward price for the delivery at \( t \) of a discount bond maturing at \( t+\tau \) with face value of $1. Thus, \( F_t(0) = d_{t+\tau}(0) \) and so \( d_{t,t+\tau}(0) = d_{t+\tau}(0) /d_t(0) \) which is the stated relation.

F. BKM, Chapter 23, Question 8.

Since the forward price is higher than implied by covered interest parity (say $1.58/£), want to buy the synthetic 1-year $-denominated discount bond and sell the 1-year $-denominated
discount bond:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell $1 of 1-year $-denominated discount bonds at time 0 and close out at 1</td>
<td>$1</td>
<td>-$1.04</td>
</tr>
<tr>
<td>Sell a forward contract at 0 which delivers [1/1.6][1+0.08] = 0.675 £ at time 1</td>
<td>0</td>
<td>[1.58-£1] x 0.675 = $1.58 x 0.675 - £0.675</td>
</tr>
<tr>
<td>Buy $1 worth of £ at 0 (£1/1.6) and invest in 1-year £-denominated discount bonds and hold til maturity.</td>
<td>-$1</td>
<td>£ [1/1.6][1+0.08] = £0.675</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>0</td>
<td>1.0665 - 1.04 = 0.0265</td>
</tr>
</tbody>
</table>

This strategy is an arbitrage opportunity.