
I. Reading.
   A. BKM, Chapter 20, Section 20.4.
   B. BKM, Chapter 21, ignore Section 21.3 and skim Section 21.5.

II. Preliminaries.
   A. Up until now, we have been concerned with the payoffs of put and call options at maturity. This handout is concerned with:
      1. The value of a call or put option prior to maturity.
      2. Whether an American call option or an American put option should be exercised prior to maturity.
   B. The results in this handout refer to non-dividend paying underlying assets unless otherwise stated.
   C. Notation.
      1. $S(0)$ be the value of the underlying at time 0.
      2. $d_T(0)$ be the discount factor for a $T$-year discount bond available at time 0.
      3. $C_{X,T}(0)$ be the time 0 price of a European call option with exercise price $X$ and expiration date $T$.
      4. $c_{X,T}(0)$ be the time 0 price of an American call option with exercise price $X$ and expiration date $T$.
      5. $P_{X,T}(0)$ be the time 0 price of a European put option with exercise price $X$ and expiration date $T$.
      6. $p_{X,T}(0)$ be the time 0 price of an American put option with exercise price $X$ and expiration date $T$. 
III. No Arbitrage Pricing Bounds.
   A. Call Options.
      1. Floor on the Value of a European Call.
         a. Have already established that a European call option must have a nonnegative price since its cashflow at expiration is nonnegative: \( C_{X,T}(0) \geq 0 \).
         b. Consider the following two investment strategies:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy underlying at 0 and sell at T</td>
<td>-S(0)</td>
<td>S(T)</td>
</tr>
<tr>
<td>Sell a T period discount bond with face value of 50 and hold to maturity</td>
<td>50 d_T(0)</td>
<td>-50</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>-S(0) +50 d_T(0)</td>
<td>S(T)-50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy a European call option at 0 with exercise price of 50 and hold until expiration at T</td>
<td>-C_{50,T}(0)</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>S(T)-50</td>
</tr>
</tbody>
</table>
c. Can see that the second strategy always produces a cash flow equal to or greater than the first strategy:
   (1) If $S(T) \geq 50$, both strategies generate the same cash flow at $T$: $[S(T) - 50]$.
   (2) If $S(T) < 50$, the first strategy generates $[S(T) - 50]$ while the second strategy generates 0 which is $>[S(T) - 50]$.

d. For there not to exist an arbitrage opportunity, the second strategy must cost more than the first one; i.e., $C_{50,T}(0) \geq S(0) - 50 d_T(0)$.
   This restriction is a floor on the call’s value.

e. So more generally, $C_{X,T}(0) \geq S(0) - X d_T(0)$ and $C_{X,T}(0) \geq 0$ which implies that $C_{X,T}(0) \geq \max\{S(0) - X d_T(0), 0\}$.
f. Example. Bloomberg screen for 4/15/97 Microsoft options. May 97 options expire 5/17/97. Microsoft is not paying any dividend between 4/15/96 and 5/17/96. The “Fin Rate” is the riskfree rate and is 5.28%. Assuming this is a continuously compounded annual rate, it implies a discount factor on a 32-day discount bond of \( e^{0.0528\times32/365} = 0.9954 \).

(1) For \( X=85 \),
\[
\max\{S(0) - X d_t(0), 0\} = \max\{98.375\times0.9954, 0\} = \max\{13.766, 0\} = 13.766 < 14.5
\]
which is the ask price of the call.

(2) Suppose the ask price of the 85 call on 4/15/97 is 13.5 which violates the floor of 13.766. There is an arbitrage opportunity which involves buying the call (since it is undervalued) and selling the first strategy described above:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>4/15/97</th>
<th>5/17/97</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S(5/17/97)&lt;85</td>
<td>S(5/17/97)≥85</td>
</tr>
<tr>
<td>Buy 1 Msft call option on 4/15/97 with exercise price of 85 and hold until expiration on 5/17/97</td>
<td>-13.5</td>
<td>0</td>
</tr>
<tr>
<td>Sell 1 Msft share on 4/15/97 and close out on 5/17/97</td>
<td>98.375</td>
<td>-S(5/17/97)</td>
</tr>
<tr>
<td>On 4/15/97, buy a discount bond with face value of 85 maturing on 5/17/97 and hold to maturity</td>
<td>-85 \times 0.9954 = 84.609</td>
<td>85</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>0.266</td>
<td>85-S(5/17/97)&gt;0</td>
</tr>
</tbody>
</table>

On 4/15/97, buy a discount bond with face value of 85 maturing on 5/17/97 and hold to maturity.

\( 85 - 84.609 = 0.391 \)
2. Early Exercise of an American Call.
   a. Know that an American option must be worth at least as much as a European option with the same expiry date and exercise price: \( c_{X,T}(0) \geq C_{X,T}(0) \).
   b. So the American call has the same floor as the European call.
   c. The value of a European option with same expiry date and exercise price can be thought of as the value of holding the American option til the exercise date.
   d. If this value exceeds the value of exercising the American option now, and this is true for any date prior to \( T \), then it is optimal to hold the American option til maturity.
   e. In general, the floor on the value of the call associated with not exercising prior to maturity is always greater than or equal to the value of exercising the call now (since \( d_T(0) < 1 \):
      \[
      c_{X,T}(0) \geq \max\{S(0) - X d_T(0), 0\} \geq \max\{S(0) - X, 0\}.
      \]
   f. So the holder of an American call option on a non-dividend paying underlying never wants to exercise early.
### B. Put Options.

1. **Floor on the Value of a European Put.**

   a. Consider the following two investment strategies:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell underlying at 0 and close out at T</td>
<td>S(0)</td>
<td>-S(T)</td>
</tr>
<tr>
<td>Buy a T period discount bond with face value</td>
<td>-50 d_r(0)</td>
<td>50</td>
</tr>
<tr>
<td>of 50 and hold to maturity</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>S(0) - 50 d_r(0)</td>
<td>50 - S(T)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy a European put option at 0 with</td>
<td>-P_{50,T}(0)</td>
<td>[50 - S(T)]</td>
</tr>
<tr>
<td>exercise price of 50 and hold until</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>expiration at T</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Can see that the second strategy always produces a cash flow equal to or greater than the first strategy:

1. If \( S(T) < 50 \), both strategies generate the same cash flow at \( T: \) \([50 - S(T)]\).
2. If \( S(T) \geq 50 \), the first strategy generates \([50 - S(T)]\) while the second strategy generates 0 which is \( \geq [50 - S(T)] \).

For there not to exist an arbitrage opportunity, the second strategy must cost more than the first one; i.e., \( P_{50,T}(0) \geq 50 d_T(0) - S(0) \).

Combining this floor with the nonnegativity restriction, obtain:

\[
P_{50,T}(0) \geq \max\{50 d_T(0) - S(0), 0\}.
\]

For general exercise price \( X \), obtain the following floor:

\[
P_{X,T}(0) \geq \max\{X d_T(0) - S(0), 0\}.
\]


1. For \( X=85 \),
\[
\max\{X d_T(0) - S(0), 0\} = \max\{85 \times 0.9954-98.5, 0\} = \max\{-13.891, 0\} = 0 < 0.875
\]
which is the ask price of the put.

2. For \( X=105 \),
\[
\max\{X d_T(0) - S(0), 0\} = \max\{105 \times 0.9954-98.5, 0\} = \max\{6.017, 0\} = 6.017 < 8.5
\]
which is the ask price of the put.
2. Early Exercise of an American Put.
   a. Know that an American option must be worth at least as much as a European option with the same expiry date and exercise price: 
      \[ p_{X,T}(0) \geq P_{X,T}(0). \]
   b. The value of a European option with same expiry date and exercise price can be thought of as the value of holding the American option till the exercise date.
   c. In general, the floor on the value of the put associated with not exercising prior to maturity is less than the value of exercising the put now (since \( d_T(0) < 1 \)):
      \[ p_{X,T}(0) \geq \max \{ X - S(0), 0 \} \geq \max \{ X d_T(0) - S(0), 0 \}. \]
   d. So the holder of an American put option on a non-dividend paying underlying may want to exercise early.
   e. In fact, it can be shown that for \( S(0) \) sufficiently small, the holder of an American put on a non-dividend paying underlying prefers to exercise immediately.
C. Put Call Parity.

1. Consider the following two investment strategies:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy underlying at 0 and sell at T</td>
<td>-S(0)</td>
<td>S(T)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time 0</th>
<th>Time T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy a European call option at 0 with</td>
<td>-C_{50,T}(0)</td>
<td>0</td>
</tr>
<tr>
<td>exercise price of 50 and hold until</td>
<td></td>
<td>S(T)</td>
</tr>
<tr>
<td>expiration at T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Write a European put option at 0 with</td>
<td>P_{50,T}(0)</td>
<td>-[50-S(T)]</td>
</tr>
<tr>
<td>exercise price of 50 and hold until</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>expiration at T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy a T period discount bond with</td>
<td>-50 d_t(0)</td>
<td>50</td>
</tr>
<tr>
<td>face value of 50 and hold to maturity</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>Net Cash Flow</td>
<td>-C_{50,T}(0)+P_{50,T}(0)</td>
<td>S(T)</td>
</tr>
<tr>
<td></td>
<td>-50 d_t(0)</td>
<td>S(T)</td>
</tr>
</tbody>
</table>

2. Can see that these two strategies have the same cash flows. The law of one price says that these strategies must have the same price.

3. Thus, get a relation between the price of a European call and put with the same exercise date and price and the price of the underlying and the present value of the exercise price:

\[ S(0) = C_{50,T}(0) - P_{50,T}(0) + 50 d_t(0). \]

4. For general exercise price X,

\[ S(0) = C_{X,T}(0) - P_{X,T}(0) + X d_t(0). \]

5. This relation is known as put call parity.

6. Can also see the relation using payoff diagrams. Sum the payoff diagrams for the second strategy and you get the payoff at T from holding the underlying.

7. Example. Msft price today is 98.385. The price of a discount bond (face value of 100) maturing in 32 days is 99.54. A European call expiring in 32 days with an exercise price of 85 has a price of 14.5 today. What is the price today of a European put expiring in 32 days with an exercise price of 85?

\[
P_{85,32\text{day}}(0) = C_{85,32\text{day}}(0) - S(0) + 85 d_{32\text{day}}(0) = 14.5 - 98.375 + 85 x 0.9954 = 0.359.
\]
Long 1 European Call expiring at $T$ with $X=S_0$, Short 1 European Put expiring at $T$ with $X=S_0$ and Long a Discount Bond maturing at $T$ with face value of $S_0$; Payoff at $T$

Long 1 Stock; Payoff at $T$

Payoff at $T$
IV. Black Scholes Model.

A. Assumptions.
1. Yield curve is flat through time at the same interest rate. So there is no interest rate uncertainty.
2. Underlying asset return is lognormally distributed with constant volatility and does not pay dividends.
3. Continuous trading is possible.
4. No transaction costs, taxes or other market imperfections.

B. Notation. Above notation holds. Additionally,
1. \( r' \) is the continuously compounded annual riskfree rate. So $1 invested today at the riskless rate is worth \( S(0) e^{r't} \) in \( t \) years time.
2. \( \sigma \) is the volatility of the continuously compounded annual return on the underlying asset.

C. Formula for European Call Options.
1. The value of a European call option is given by:
\[
C_{X,T}(0) = S(0) N(d_1) - X e^{-r'T} N(d_2)
\]
where \( N(.) \) is the cumulative Normal distribution function (see BKM, Table 20.2);
\[
d_1 = \frac{\ln[S(0)/X] + \{r' + \sigma^2/2\} T}{\sigma \sqrt{T}}; \quad d_2 = d_1 - \sigma \sqrt{T}.
\]
2. Factors affecting the value of the call.
   a. \( S(0) \): \( C_{X,T}(0) \) is monotonically increasing in \( S(0) \) as would be expected.
   b. \( X \): \( C_{X,T}(0) \) is monotonically decreasing in \( X \) as would be expected.
   c. \( \sigma \): \( C_{X,T}(0) \) is monotonically increasing in \( \sigma \). Why?
      (1) Option feature of the call truncates the payoff at 0 when the underlying’s value is less than the strike price.
      (2) When \( \sigma \) increases, the volatility of \( S(T) \) increases.
      (3) The call option holder benefits from the greater upside potential of \( S(T) \) but does not bear the greater downside potential due to the truncation of the option payoff at 0.
      (4) Thus, the value of the call increases relative to \( S(0) \).
   d. \( T \): \( C_{X,T}(0) \) is monotonically increasing in \( T \). Why?
      (1) The exercise price does not have to be paid until time \( T \). When \( T \) increases, the current value of \( X \) paid at \( T \) decreases making the option more valuable for given \( S(0) \).
      (2) Second, with a longer time to maturity the volatility of \( S(T) \) increases for given \( \sigma \). So the value of the call today increases for the same reason that an increase in \( \sigma \) increases the call’s value today.
      (3) Both effects are acting in the same direction.
   e. \( r' \): \( C_{X,T}(0) \) is monotonically increasing in \( r' \). Why?
      (1) The exercise price does not have to be paid until time \( T \). When \( r' \) increases, the current value of \( X \) paid at \( T \) decreases making the option more valuable for given \( S(0) \).

3. Factors not affecting the value of the call.
   a. the expected return on the underlying asset. Why?
      (1) When the expected return on the underlying increases, the expected return on the call also increases.
      (2) Since the current underlying’s price \( S(0) \) remains equal to the current value of the underlying despite its higher expected return, \( C(0) \) remains the current value of the call option despite its higher expected return.
D. Value of an American call option.
1. It was shown above that the holder of an American call option on a non-dividend paying asset would never exercise early.
2. Thus, the value of an American call equals the value of the European call with the same exercise price and date.
3. So the value of an American call is also given by the Black Scholes call option formula.
E. European Put Options.
1. Once the value of the European call with same exercise price and date has been determined, put call parity can be used to determine the value of a European put.
2. Factors affecting the value of the European put.
   a. $S(0)$: $P_{X,T}(0)$ is monotonically decreasing in $S(0)$ as would be expected.
   b. $X$: $P_{X,T}(0)$ is monotonically increasing in $X$ as would be expected.
   c. $\sigma$: $P_{X,T}(0)$ is monotonically increasing in $\sigma$. Why?
      (1) Option feature of the put truncates the payoff at 0 when the underlying’s value is more than the strike price.
      (2) When $\sigma$ increases, the volatility of $S(T)$ increases.
      (3) The put option holder benefits from the greater downside potential of $S(T)$ but does not bear the greater upside potential due to the truncation of the option payoff at 0.
      (4) Thus, the value of the put increases relative to $S(0)$.
   d. $T$: affect on $P_{X,T}(0)$ is ambiguous. Why?
      (1) The exercise price is not received until time $T$. When $T$ increases, the current value of $X$ received at $T$ decreases making the option less valuable for given $S(0)$.
      (2) But acting in the other direction, a longer time to maturity increases the volatility of $S(T)$ for a given $\sigma$. When $T$ increases, the value of the put today also increases for the same reason that an increase in $\sigma$ (for fixed $T$) increases the put’s value today.
      (3) It is not clear which of these effects dominates.
   e. $r'$: $P_{X,T}(0)$ is monotonically decreasing in $r'$. Why?
      (1) The exercise price is not received until time $T$. When $r'$ increases, the current value of $X$ received at $T$ decreases making the option less valuable for given $S(0)$.

F. American Put Options.
1. No closed form solution exists to the value of an American put option.
2. A value can be obtained numerically.

G. Implied Volatility.
1. All the inputs into the Black-Scholes model are readily observable except the volatility of the return on the underlying.
2. The value of $\sigma$ which together with the other Black-Scholes inputs gives a Black-Scholes call price equal to the market’s prevailing call price is known as the implied volatility of the underlying.
<table>
<thead>
<tr>
<th>States are Equally Likely</th>
<th>Stock $S(t)$</th>
<th>Call $C_{50,T(t)}$</th>
<th>Put $P_{50,T(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff at T - State 1</td>
<td>40</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>Payoff at T - State 2</td>
<td>60</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>E[Payoff at T]</td>
<td>50</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$\sigma$[Payoff at T]</td>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>States are Equally Likely</th>
<th>Stock $S(t)$</th>
<th>Call $C_{50,T(t)}$</th>
<th>Put $P_{50,T(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff at T - State 1</td>
<td>20</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>Payoff at T - State 2</td>
<td>80</td>
<td>30</td>
<td>0</td>
</tr>
<tr>
<td>E[Payoff at T]</td>
<td>50</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>$\sigma$[Payoff at T]</td>
<td>30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>States are Equally Likely</th>
<th>Stock $S(t)$</th>
<th>Call $C_{50,T(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff at T - State 1</td>
<td>20</td>
<td>25</td>
</tr>
<tr>
<td>Payoff at T - State 2</td>
<td>80</td>
<td>85</td>
</tr>
<tr>
<td>E[Payoff at T]</td>
<td>50</td>
<td>55</td>
</tr>
<tr>
<td>$\sigma$[Payoff at T]</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Price(0)</td>
<td>40</td>
<td>10</td>
</tr>
<tr>
<td>E[Return]</td>
<td>25%</td>
<td>37.5%</td>
</tr>
</tbody>
</table>

### Price(0) and E[Return] Calculation

- For the first scenario: $E[Return] = \frac{50 - 40}{40} = 25\%$
- For the second scenario: $E[Return] = \frac{50 - 40}{40} = 25\%$
- For the third scenario: $E[Return] = \frac{50 - 40}{40} = 25\%$
- For the fourth scenario: $E[Return] = \frac{50 - 40}{40} = 25\%$