Lecture Notes 2

Concepts and Tools for Portfolio, Equity Valuation, Fixed Income, and Derivative Analyses:

Applications to Savings/CDs, Loans/Mortgages

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III. Time Line
IV. Investing for a Single Period
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Buzz Words: Time Value of Money, Equilibrium, Arbitrage, Perfect Creditworthiness (No Credit Risk, No Default Risk), Locked-in Rates, Discounting, Compounding, EAR, APY, APR, Continuous Compounding, Basis Point, Zeros.
I. Readings

RWJ Chapters 4 and 5.

Web:  http://www.fidelity.ca/fidelity/cda/ext_app/growth_calculator_eng/

Investment Growth Calculator: treat the “Rate of Return” as an interest rate, and enter values to visualize the power of compounding.

II. What is an Interest Rate?

A. The time value of any commodity reflects:
   1. Preferences for consumption sooner than later.
   2. Physical productivity.
   3. “Convenience yield” (inventories).

B. The time value of money reflects that:
   1. Money can be converted into consumption.
   2. Money can be converted into physical capital for production.
   3. Money’s convenience yield stems from the value of cash in facilitating transactions (it is a medium of exchange).

C. In “equilibrium,” these characteristics of money are valued equally, and hence they define the real rate of interest.

D. Inflation

1. Inflation measures the rate of depreciation of money.

   Inflation implies a distinction between real interest rate (measured in commodities) and nominal interest rate (measured in currency).

   Example: If the productivity of physical capital is 4% (a real interest rate), and inflation is 3%, the nominal interest rate is (approximately) 7%.

2. All the interest rates in this course are nominal.
III. Time Line

$1 received today is not the same as a $1 received in the future.

The timing of a cash flow affects its value.

$1 today is never worth less than $1 in the future

When valuing cash-flow streams, the timing of the cash flows is crucial; a good idea is to draw a time line:

Money today:

\[
t=0 \quad \begin{array}{c}
\ldots \\
$100 \\
\ldots \\
\end{array}
\]

\[
t=0 \quad \begin{array}{c}
\ldots \\
$100 \\
\ldots \\
\end{array}
\]

is NOT the same as Money in the future.

The latter time line represent the contractual payment of a debt contract. Specifically, this can be a bond, that promises to pay its owner a terminal single cash flow. Since there are no intermediate cash flows (called coupon payments), this is a zero-coupon bond, or simply a “zero.”

If $1 in the future, is worth more than $1 today, you can arbitrage (“free lunch”). In well functioning markets, any arbitrage is quickly eliminated, and so for our purposes, we assume there are no arbitrage opportunities.
IV. Investing for a Single Period

A. Definition

The effective interest rate \( r \) (expressed as a decimal) over any period tells what \( x \) will be worth at the end of the period using the following formula:

\[
x (1+r)
\]

B. Example

Suppose you can invest $100 at an effective annual interest rate of 12%, by buying a CD (certificate of deposit). What is your $100 worth at the end of the year?

Answer:
First, we draw the Time Line, using the notation \( C \) for cash flow, and using subscripts for the timing of the cash flow on the time line:

\[
\begin{array}{c|c}
0 & \text{end of year 1} \\
\hline
$100 & C_1
\end{array}
\]

Next, we perform the calculations:

\[
C_0 = $100 \\
Interest = $100 \times 0.12 = $12 \\
C_1 = C_0 + Interest = $100 + $12 = $112
\]

\[
\begin{array}{c|c}
0 & \text{end of year 1} \\
\hline
$100 & $112
\end{array}
\]
V. Single Cash Flow, Multiple Periods, and Future Value

A. Example

Suppose you can invest $100 at an effective annual interest rate of 12%, by investing in your offshore savings account. What is your $100 worth at the end of 3 years?

1. One Way to Answer: Obtain $C_3$ in 3 steps

   \[
   \begin{align*}
   C_0 &= $100 \\
   C_1 &= C_0 (1+r) = 100(1+0.12) = $112 \\
   C_2 &= C_1 (1+r) = 112(1+0.12) = $125.44 \\
   C_3 &= C_2 (1+r) = 125.44(1+0.12) = $140.49
   \end{align*}
   \]

   \[
   \begin{array}{cccc}
   0 & 1 & 2 & 3 \\
   $100 & $112 & $125.44 & $140.49 \\
   \end{array}
   \]

2. Another Way to Answer: The 3 steps could be combined into 1 step

   \[
   C_0 = $100
   \]
   \[
   C_3 = C_0 (1+r)^3 = 100(1+0.12)^3 = $140.49
   \]

   More conveniently:
   \[
   C_3 = C_0 (1+r)^3 = $100(1+0.12)^3 = $140.49
   \]

3. First, notice that we reinvest interest to earn more interest. This is referred to as compounding.

   Second, note how the answer above is the initial investment $C_0$ times a multiplier that only depends on the effective interest rate and the investment interval. This multiplier is known as the future value interest factor (FVIF).
Foundations of Finance: Concepts and Tools for Portfolio, Equity Valuation, Fixed Income, and Derivative Analyses: Applications to Savings/CDs, Loans/Mortgages

B. The future value (FV) formula answers the following question:

If we invest some money at a given effective interest rate, how much money would we have at some future time?

Specifically, if we invest [borrow] \( C_n \) in \( n \) periods (\( n=0 \) means “today”) at a given effective interest rate per period of \( r \) (expressed as a decimal), how much money would we have [owe] after \( t \) periods from investing [borrowing] the money; i.e., what is \( C_{n+t} \)?

C. The future (terminal) value formula:

\[
C_{n+t} = C_n (1+r)^t
\]

where \((1+r)^t = FVIF(r,t)\) is the future value interest factor.

Notice that the future value interest factor does not depend on when the money is invested.

D. Example

Suppose you can lend $100 in one year's time at an effective annual interest rate of 12%, without any default risk. What is your $100 worth after being repaid for 3 years?

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 \\
$100 & & & & C_4 \\
\end{array}
\]

Answer: Use the future value formula. \( C_1 = $100 \)

\[
C_4 = C_1 \times FVIF(0.12,3) = C_1 (1+0.12)^3 = $100(1+0.12)^3 = $140.49
\]
VI. Single Cash Flow, Multiple Periods, and Present Value

A. Example

Suppose you can invest at an effective annual interest rate of 12%. How much do you need to invest today to have $140.49 at the end of 3 years?

Answer: Use the future value formula, which tells you that

\[ C_3 = C_0 (1+r)^3 = \$100(1+0.12)^3 = \$140.49 \]

this implies

\[ C_0 = \frac{C_3}{(1+r)^3} = \frac{\$140.49}{(1+0.12)^3} = \$100 \]

Notice how the answer is the final value \( C_3 \) times a multiplier that only depends on the effective interest rate and the investment interval. This multiplier is known as the present value interest factor (PVIF), or as the discount factor.

We often refer to \( C_0 \) as the present value (PV):

\[ \$100 = PV(\$140.49, 12\%, 3) \]

On a typical Financial Calculator: 140.49 FV, 12 I/YR, 3 N, PV

The calculator displays: -99.998. Why negative? Why not 100?
B. The present value (PV) formula answers the following question:

If we can invest money at a given effective interest rate, how much money do we need to invest today to have a given sum at some future time?

Specifically, if we can invest [borrow] money at a given effective interest rate \( r \) (expressed as a decimal), how much money, \( C_n \), do we need to invest [borrow] in \( n \) periods \( (n=0 \text{ means “today”}) \) to have [repay] a given sum \( C_{n+t} \) in \( (n+t) \) periods?

C. The present value formula:

\[
C_n = C_{n+t} \frac{1}{(1 + r)^t}
\]

where \( 1/(1+r)^t = (1+r)^{-t} = \text{PVIF}(r,t) \) is the present value interest factor.

D. Example

Suppose you can invest at an effective annual interest rate of 12%. How much do you need to invest at the start of next year to have $140.49 at the end of 4 years from today?

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td></td>
<td></td>
<td></td>
<td>$140.49</td>
</tr>
</tbody>
</table>

\( C_1 \)

\[
C_1 = C_4 \times \text{PVIF}(0.12,3) = C_4 / (1+r)^3 = $140.49 / (1+0.12)^3 = $100
\]

Answer: Use the present value formula: \( C_4 = $140.49 \)
VII. Equivalent Effective Interest Rates Over Different Compounding Periods.

A. Example

Suppose the effective annual interest rate is 12%. What is the effective 3 year interest rate?

Answer: We saw already that

\[ C_3 = C_0 \times (1+r)^3 = 100(1+0.12)^3 = 100(1+0.4049) = 140.49 \]

This implies, using the definition of effective rate, that the effective 3-year rate is 40.49%.

Note that: Effective 3-year rate = 40.49% = 3 \times 12\% = 3 \times \text{Effective 1-year rate.}

B. The Effective Rate Formula (the relation between the effective rates for compounding periods of different lengths):

The effective t-period rate \( r_t \) (expressed as a decimal) is related to the effective one period rate \( r \) by (note that \( t \) can be a fraction of a period)

\[ (1+r_t) = (1+r)^t \]

C. Example

Suppose the effective monthly rate is 0.94888%. What is the Effective Annual Rate (EAR)?
Answer: Here one period is a month, so \( t = 12 \), and \( r = 0.0094888 \).
Using the effective rate formula

\[
1 + r_{12} = (1 + r)^{12} = 1.0094888^{12} = 1.12
\]

and so \( r_{12} = 0.12 \) and the effective annual rate (EAR) is 12%. EAR is also often referred to as Annual Percentage Yield (APY).

D. Fractional holding periods

For example, if the effective 1-year rate is known, the monthly effective rate can be calculated by using the formula with \( t = 1/12 \)

Example
Suppose the effective annual rate (EAR) is 12%. What is the effective monthly rate?

<table>
<thead>
<tr>
<th>today</th>
<th>end of year 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1/12</td>
<td>2/12</td>
</tr>
<tr>
<td>3/12</td>
<td>4/12</td>
</tr>
<tr>
<td>5/12</td>
<td>6/12</td>
</tr>
<tr>
<td>7/12</td>
<td>8/12</td>
</tr>
<tr>
<td>9/12</td>
<td>10/12</td>
</tr>
<tr>
<td>11/12</td>
<td>1</td>
</tr>
</tbody>
</table>

Answer: Here one period is a year, so \( r = 0.12 \). We want to compute the effective rate over a shorter period than a year, specifically, \( t = 1/12 < 1 \). Using the effective rate formula:

\[
1 + r_{1/12} = (1 + r)^{1/12} = 1.12^{1/12} = 1.009488
\]

and so \( r_{1/12} = 0.009488 \), and the effective monthly rate is 0.9488%.

Note: When \( t < 1 \), you may find it more intuitive to rewrite the relation between the effective rates for compounding periods of different lengths \((1 + r_t) = (1 + r)^t\), given on the previous page, by raising both sides of the equation to the power \( 1/t \):

\[(1 + r_t)^{1/t} = (1 + r).
\]

In the above example \((1 + r_{1/12})^{1/(1/12)} = (1 + 0.12)\), and so \(1 + r_{1/12} = 1.009488\)
VIII. Alternate Interest Rate Concepts

A. Annual Percentage Rate (APR)

When the compounding period (over which the interest “gets added to the account”) is some fraction of a year, e.g., $1/M$, instead of quoting the interest rate per $1/M$ of a year, i.e., the effective $1/M$-year rate (expressed as a decimal) $r_{1/M}$, the rate that financial institutions often quote (and some are required to quote) is in the form of a product: $r_{1/M} \times M$.

This rate is known as the annual percentage rate (APR); i.e., $\text{APR} = r_{1/M} \times M$. The APR is a mere quoting device. It does not capture the intraperiod compounding. Only for annual compounding, i.e., $M=1$, we have $\text{APR}=\text{EAR}$.

B. Example

You borrow $100 at an APR of 12%. What is the EAR when the compounding is semiannual, quarterly, or monthly, respectively?

Answer: We have three cases, where one period is one half year, one quarter, or one month. Using the definition of APR, and the effective rate formula:

1. With semiannual compounding ($M=2$): $FV = 100 \times (1.06)^2 = 112.36$
   EAR = 12.36%

2. With quarterly compounding ($M=4$): $FV = 100 \times (1.03)^4 = 112.55$
   EAR = 12.55%

3. With monthly compounding ($M=12$): $FV = 100 \times (1.01)^{12} = 112.68$
   EAR = 12.68%

C. Formulas.

We keep denoting the EAR as $r$, so $r_{1/M} = (1+r)^{1/M} - 1 = \text{APR} / M$

1. From APR to EAR (both expressed as decimals): $r = (1 + \frac{\text{APR}}{M})^M - 1$

2. From EAR to APR (both expressed as decimals): $\text{APR} = [(1 + r)^{\frac{1}{M}} - 1] M$
D. Continuous Compounding

1. **Definition:**

   Let \( r \) be the effective interest rate for one period. Define \( \rho \), the *continuously compounded interest rate* for a period as follows:

   \[
   e^{\rho} = (1+r)
   \]

   The continuous rate is obtained from the effective rate as follows:

   \[
   \rho = \ln (1+r).
   \]

*Interpretation of the continuously compounded interest rate:*

Everybody knows that

\[
e^{\rho} = \lim_{M \to \infty} (1+\frac{\rho}{M})^M = (1+r)
\]

So continuously compounded annual rate, \( \rho \), can be interpreted as the APR associated with an infinitesimally small compounding period (instantaneous compounding).

2. **Example**

   The effective annual rate is 12%. What is the continuously compounded annual rate?

   Answer: \( r = 0.12 \). The continuously compounded annual rate (expressed as a decimal) is \( \rho = \ln (1+r) = \ln (1.12) = 0.1133 \).
3. **Rules for using continuous interest rates**

   a) The continuous interest rate for any period of length \( t \) (in annual units) is given by \( \rho_t = \rho \times t \) (recall that this is not true for effective interest rates \( r_t \neq r \times t \)). This additivity over time is a major advantage of using continuous compounding. **Option Pricing** formulas often use continuously compounded rates.

   b) The future value formula:
   \[
   C_{n+t} = C_n \times FVIF(\rho, t) = C_n (1+\rho)^t = C_n e^{\rho t}
   \]

   c) To obtain present values:
   \[
   C_n = C_{n+t} \times PVIF(\rho, t) = C_{n+t} (1+\rho)^{-t} = C_{n+t} e^{-\rho t}
   \]

4. **Example**

   The effective annual rate is 12%. What is the continuously compounded semiannual rate? How much will $500 invested today be worth in 6 months?

   **Answer**: The continuously compounded annual rate is 11.33%. Since \( \rho_{1/2} = \rho(1/2) = 0.1133 \times 0.5 = 0.05666 \), the continuously compounded semiannual rate is 5.666%.

   There are two approaches to obtaining what $500 will be worth in 6 months [same end result, because FVIF(\( \rho, t \)) = (1+\( \rho \))^t = e^{\rho t}, and which rate we like to use is just a matter of convenience]

   a) **Continuous Compounding**:
   using the FV formula for continuous compounding (with \( n=0, t=1/2 \))
   \[
   C_{1/2} = $500 e^{\rho_{1/2}} = $500 e^{0.05666} = $500 \times 1.0583005 = $529.15
   \]

   b) **Discrete Compounding**:
   \[
   C_{1/2} = $500(1+r)^{1/2} =$500 \times 1.12^{1/2} =$500\times1.0583005= $529.15
   \]
IX. Multiple Cash Flows.

A. Rules

1. Cash flows occurring at different points in time cannot be added. (Only cash flows which occur at the same time can be added.)

   How much do you need to invest today to cover these two obligations?

   \[
   \begin{array}{cccc}
   & 0 & 1 & 2 & 3 \\
   C_0 & \$500 & \$800 & & \\
   r & 10\% & & & \\
   \end{array}
   \]

   \[
   \text{PVIF}(0.1,2) = \$500 \left(1 + 0.1\right)^{-2} = \$413.22
   \]

   \[
   \text{PVIF}(0.1,3) = \$800 \left(1 + 0.1\right)^{-3} = \$601.05
   \]

   Total amount needed to be invested today is

   \[
   C_0 = \$413.22 + \$601.05 = \$1014.27
   \]

2. Once each of a set of cash flows at different points in time has been converted to a cash flow at the same point in time, those cash flows can be added to get the value of the set of cash flows at that point.

   \[
   \begin{align*}
   \$500 & \text{ PVIF}(0.1,2) = \$500 \left(1 + 0.1\right)^{-2} = \$413.22 \\
   \$800 & \text{ PVIF}(0.1,3) = \$800 \left(1 + 0.1\right)^{-3} = \$601.05
   \end{align*}
   \]

   Total amount needed to be invested today is

   \[
   C_0 = \$413.22 + \$601.05 = \$1014.27
   \]

3. The value of the stream of cash flows at any other point can then be obtained using the present or future value formulas.

   If you delay your investment date by 1 year, you will need to invest

   \[
   C_1 = \$1014.27 \ \text{ FVIF}(0.1,1) = \$1014.27 \left(1 + 0.1\right) = \$1115.70
   \]
X. A Particular Cash Flow Pattern: Annuity

$t$ equal payments made at equal intervals are referred to as an **annuity**.

<table>
<thead>
<tr>
<th>n-1</th>
<th>n</th>
<th>n+1</th>
<th>n+2</th>
<th>n+3</th>
<th>n+4</th>
<th>n+t-1</th>
<th>n+t</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<tr>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
</tr>
</tbody>
</table>

A. Converting to a single cash flow

1. Convert each cash flow to a single cash flow at a chosen point in time using the present (or future) value formula and then add up these sums to give the annuity's value at that point in time.

2. A short-cut:
If the annuity’s first cash flow occurs at $(n+1)$ and its last at $(n+t)$, use the **annuity present value factor**, $\text{APVF}(r,t)$, which satisfies

$$C_n = C \times \text{APVF}(r,t)$$

where

a) a period is equal to the interval between cash flows
b) the effective interest rate over a period is $r$
c) $C_n$ is the “single cash flow equivalent” at the point in time one period before the first cash flow
d) and $\text{APVF}(r,t)$ (also often denoted $\text{PVA}_r^t$) is

$$\text{APVF}(r,t) = \frac{1-(1+r)^t}{r}$$
5. Similarly, for the future value use the \textit{annuity future value factor}, \( AFVF(r,t) \), which satisfies

\[ C_{n+t} = C \times AFVF(r,t) \]

where

\( a) \) \( C_{n+t} \) is the “single cash flow equivalent” at the point in time of the last cash flow.

\( b) \) \( AFVF(r,t) \) (also often denoted \( FVA_r^t \)) is

\[ AFVF(r,t) = \frac{(1+r)^t - 1}{r} \]

4. \textit{How to value an annuity, which specifies the cash flow to start immediately?}
B. Example

Suppose you receive $1000 at the end of each year for the next 3 years. If you can invest at an effective annual rate of 10%, how much would you have in 4 years time?

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 & 4 \\
\$1000 & \$1000 & \$1000 & \$C_4 \\
\end{array}
\]

1. One Answer: using the future value formula

\[
\begin{align*}
$1000 (1+0.1)^3 &= $1331 \\
$1000 (1+0.1)^2 &= $1210 \\
$1000 (1+0.1)^1 &= $1100 \\
\end{align*}
\]

\[C_4 = $1331 + $1210 + $1100 = $3641\]

2. Another Way to Answer: using the future value annuity formula gives \(C_3\) and then can use the future value formula to get \(C_4\)

\[
\begin{align*}
C_3 &= C \times \text{AFVF} (0.1, 3) = $1000 \times [(1+0.1)^3-1]/0.1 = $3310 \\
C_4 &= C_3 \times \text{FVIF} (0.1, 1) = C_3 (1+0.1) = $3310 \times 1.1 = $3641 \\
\end{align*}
\]

3. Yet Another Way to Answer: using the present value annuity formula gives \(C_0\) and then we can use the future value formula to get \(C_4\)

\[
\begin{align*}
C_0 &= C \times \text{APVF} (0.1, 3) = $1000 \ [1-(1+0.1)^3]/0.1 = $2486.852 \\
C_4 &= C_0 \times \text{FVIF}(0.1, 4) = C_0 (1+0.1)^4 = $2486.852 \times 1.4641 \\
&= $3641 \\
\end{align*}
\]
C. Loan/Mortgage Amortization

Suppose you borrow $5000 at an APR of 18% compounded monthly. (1) If you have a three year loan and you make loan repayments at the end of each month, what is your monthly payment? (2) What is the loan balance outstanding after the first loan payment? (3) How much interest accumulates in the first month of the loan? (4) What is the loan balance outstanding after the twelfth loan payment? (5) How much interest accumulates in the twelfth month of the loan?

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & \ldots & 11 & 12 & 13 & \ldots & 35 & 36 \\
C & C & \ldots & C & C & C & \ldots & C & C \\
\end{array}
\]

$5000

Answer

1. Monthly payment:
The APR is 18% so the effective monthly rate is \((18\% / 12)=1.5\%\).
Thus, \(r = 0.015\).

\[
\begin{align*}
$5000 & = C \times \text{APVF}(0.015,36) = C \times [1-(1+0.015)^{-36}]/0.015 \\
& = C \times 27.660684 \\
\end{align*}
\]

and so

\[
C = \frac{$5000}{27.660684} = $180.76
\]
2. Loan balance outstanding at time 1 (after the first payment):

\[ \$5000 \times 1.015 - C = \$5075 - \$180.76 = \$4894.24 \]

Note! The loan balance outstanding at time 1 (after the first payment is made) is equal to the single cash-flow equivalent at time 1 of the remaining 35 payments \( C_1 \) [2 to 36]:

\[
\begin{array}{cccccccc}
0 & 1 & 2 & \ldots & 11 & 12 & 13 & 35 & 36 \\
\hline
\$180.76 & \$180.76 & \$180.76 & \$180.76 & \$180.76 & \$180.76 & \$180.76 & \$180.76 \\
C_1 \text{ [2 to 36]} \\
\end{array}
\]

\[ C_1 \text{ [2 to 36]} = \$180.762 \times \text{APVF}(0.015,35) \]
\[ = \$180.762 \times [1 - (1+0.015)^{-35}]/0.015 = \$4894.24 \]

On a typical financial calculator:

-180.762 PMT, 0 FV, 1.5 I/YR, 35 N, PV

The calculator displays 4,894.2386

Note:
0 for FV is the convention calculators use to understand that there are no terminal payments besides the regular payment entered via PMT.

3. Accumulated interest for month 1:

\[ \$5000 \times 0.015 = \$75. \]
4. The loan balance outstanding after the twelfth loan payment:

<table>
<thead>
<tr>
<th>Time</th>
<th>Interest</th>
<th>Balance prior to Payment</th>
<th>Payment</th>
<th>Balance after Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>5000</td>
</tr>
<tr>
<td>1</td>
<td>75</td>
<td>5075</td>
<td>180.76198</td>
<td>4894.238</td>
</tr>
<tr>
<td>2</td>
<td>73.41357</td>
<td>4967.6516</td>
<td>180.76198</td>
<td>4786.8896</td>
</tr>
<tr>
<td>3</td>
<td>71.803344</td>
<td>4858.693</td>
<td>180.76198</td>
<td>4677.931</td>
</tr>
<tr>
<td>4</td>
<td>70.168965</td>
<td>4748.0999</td>
<td>180.76198</td>
<td>4567.338</td>
</tr>
<tr>
<td>5</td>
<td>68.51007</td>
<td>4635.848</td>
<td>180.76198</td>
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<tr>
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<td>3745.3179</td>
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<tr>
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<td>3801.4977</td>
<td>180.76198</td>
<td><strong>3620.7357</strong></td>
</tr>
</tbody>
</table>

The table provides a correct, but somewhat tedious way to answer the question (it is easy to implement with a spreadsheet).

*Alternatively,*

We note that similarly to what we saw in item 2, the loan balance outstanding after 12 payments is the single cash-flow equivalent at time 12 of the remaining 24 payments:

$$
\begin{align*}
C_{12} [13 \text{ to } 36] = 12 \sum_{t=13}^{36} \frac{C_t}{(1 + r)^t}
\end{align*}
$$

$$
\begin{align*}
&= 12 \times \left( \frac{180.76}{1.0795} + \frac{180.76}{1.0795^2} + \ldots + \frac{180.76}{1.0795^{24}} \right)
\end{align*}
$$

$$
\begin{align*}
&= 12 \times 3620.7357
\end{align*}
$$

$$
\begin{align*}
C_{12} &= 3620.7357
\end{align*}
$$
\[ C_{12} \text{ [13 to 36]} = 180.762 \times \text{APVF}(0.015, 24) \]
\[ = 180.762 \left[ 1 - (1 + 0.015)^{-24} \right] / 0.015 = 3620.736 \]

5. *The interest that accumulates in the twelfth month of the loan*:

To determine how much interest accumulates in the 12th month, we need to determine the loan balance outstanding after 11 payments. This balance is the single cash-flow equivalent at time 11 of the last 25 payments:

\[
\begin{array}{cccccccc}
0 & 1 & 2 & \ldots & 11 & 12 & 13 & 35 & 36 \\
\hline
\hline
& & & & & \$180.76 & \$180.76 & \$180.76 & \$180.76 \\
C_{11} \text{ [12 to 36]} & & & & & & & & \\
\hline
\end{array}
\]

\[ C_{11} \text{ [12 to 36]} = 180.762 \times \text{APVF}(0.015, 25) \]
\[ = 180.762 \left[ 1 - (1 + 0.015)^{-25} \right] / 0.015 = 3745.32 \]

So the amount of interest accumulating during the 12th month is:

\[ 3745.32 \times 0.015 = 56.180 \]
XI. A Particular Cash Flow Pattern: Perpetuity

Equal payments made at equal intervals, forever, are referred to as a *perpetuity*.

\[
\begin{array}{cccccc}
\text{n-1} & \text{n} & \text{n+1} & \text{n+2} & \text{n+3} & \text{n+4} \\
C & C & C & C & \ldots
\end{array}
\]

A. Converting to a single cash flow

When the perpetuity’s first cash flow occurs at \((n+1)\), use the *perpetuity present value factor*, \(PPVF(r)\), which satisfies

\[
C_n = C \times PPVF(r)
\]

where the effective interest rate over a period is \(r\), the formula gives the single cash-flow equivalent at the point in time one period before the first cash flow, and \(PPVF(r)\) (also often denoted \(PVP_r\)) is

\[
PPVF(r) = \frac{1}{r}.
\]

Explanation:

\[
PPVF(r) = \lim_{t \to \infty} APVF(r,t) = \lim_{t \to \infty} \frac{1 - (1 + r)^{-t}}{r} = \frac{1}{r}.
\]

*Intuitively*: To be able to pay \(C\) each period by earning the rate \(r\) on the balance, the balance (denote it \(P\)) must not change. It thus must be \(P = C/r\), so that \(P \times r = C\) can be withdrawn, while \(P\) dollars remain. Therefore, since you need \(C/r\) dollars to generate a perpetual stream of cash flow \(C\), it means that the value of such perpetual stream, if offered to you, must be also \(C/r\) (otherwise, you can arbitrage).
B. Example

Mr X wants to set aside in a trust an amount of money today that will pay his son and his descendants $10,000 at the end of each fiscal year forever, with the first payment to be made two years from today. If Mr X can invest at an effective annual rate of 10%, how much would he have to invest today?

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
C_0 & $10000 & $10000 & $10000 \\
\end{array}
\]

Answer

First, calculate how much Mr X needs to have at the end of the first fiscal year, using the formula for the present value of perpetuity. We can use this formula because the first payment is at the end of the second fiscal year and the payments are made annually.

\[
C_1 = $10000 \times \text{PPVF}(0.1) = $10000/0.1 = $100000.
\]

Second, calculate the amount that Mr X must invest today, using the present value formula:

\[
C_0 = C_1 \times \text{PVIF}(0.1,1) = $100000/(1+0.1) = $90909.
\]