XII. Appendix

This appendix is dedicated to those who love the math within finance. You may skip it if you wish.

Reminder: Geometric Progression

\[ a_1 \]
\[ a_2 = a_1 q \]
\[ a_3 = a_1 q^2 \]
\[ \vdots \]
\[ a_N = a_1 q^{N-1} \]

Find the sum \( S \), when \( q \neq 1, \ N > 1 \):

\[ S = a_1 + a_1 q + a_1 q^2 + \ldots + a_1 q^{N-1} \]

\[ qS = a_1 q + a_1 q^2 + \ldots + a_1 q^{N-1} + a_1 q^N \]

Subtract the first line, \( S \), from the second, \( qS \), to get:

\[ S(q - 1) = a_1 q^N - a_1 \]

So,

\[ S = \frac{a_1 (q^N - 1)}{q - 1} \]
The Text-Book Annuity Formula

The future value of the annuity is the sum of future values of each $c$:

$$FV_t[1:t] = c + c(1 + r) + ... + c(1 + r)^{t-1}$$

This sum is just the sum of a geometric progression, where

$$a_t = c$$
$$q = 1 + r$$
$$N = t$$

Using the above in the formula for the sum of the progression:

$$FV_t[1:t] = \frac{c((1 + r)^t - 1)}{(1 + r) - 1} = c \frac{(1 + r)^t - 1}{r}$$

The Present Value of this Annuity is then,

$$PV_0[1:t] = \frac{FV_t[1:t]}{(1 + r)^t} = c \frac{1 - (1 + r)^{-t}}{r}$$
An Annuity that Starts at $t = 0$

\[
FV_t[0 : t] = c + (1 + r) + ... + (1 + r)^{t-1} + (1 + r)^t
\]

\[a_t = c\]
\[q = 1 + r\]
\[N = t + 1\]

Use again the formula for the sum (note the different $N$):

\[
FV_t[0 : t] = \frac{c((1 + r)^{t+1} - 1)}{(1 + r) - 1} = \frac{c(1 + r)^{t+1} - 1}{r}
\]

Another way to solve:

\[
FV_t[0 : t] = FV_t[0] + FV_t[1 : t] = c(1 + r)^t + \frac{(1 + r)^t - 1}{r}
\]

\[
= \frac{c((1 + r)^t(r + 1) - 1)}{r} = \frac{c(1 + r)^{t+1} - 1}{r}
\]

The Present Value is:

\[
PV_t[0 : t] = \frac{FV_t[0 : t]}{(1 + r)^t} = \frac{c((1 + r)(1 - (1 + r)^{-(t+1)})}{r}
\]

\[= c + PV_0[1 : t] = c + \frac{c(1 - (1 + r)^{-t})}{r}\]
**How to Price a Perpetuity?**

a) As in the notes (annuity with $t = \infty$)

b) Each period the cash flow of a perpetuity “looks” the same, viewed into infinity. Hence, the price is the same (same dollar figure $P$ each period):

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>……………..</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$c$</td>
<td>$c$</td>
<td>$c$</td>
</tr>
</tbody>
</table>

So, $P$ at any period is the present value of a $c$ and a $P$:

\[
P = \frac{c + P}{1 + r}
\]

\[
P(1 + r) = c + P
\]

\[
P = \frac{c}{r}
\]

c) No arbitrage pricing

\[
\begin{array}{ccc}
\text{t = 0} & \text{t = 1} \\
\hline
\text{Borrow amount} & P & -P(1+r) \\
\text{Buy 1 perpetuity} & -P & \text{Get cash flow: } c \\
\text{Sell perpetuity: } P & & \\
\text{Total cash flow} & 0 & c - P \times r
\end{array}
\]

A zero now is worth zero in the future, so must have $c - P \times r = 0$ and,

\[
P = \frac{c}{r}
\]