Lecture Notes 5

Uncertainty, Characterizing the Return Distribution, and Investor Preferences

I. Readings and Suggested Practice Problems

II. Dealing with Uncertainty and Risk

III. How to calculate Expected Return

IV. How to calculate the Variance and Standard Deviation of Return

V. How to calculate the Covariance and Correlation between Two Returns

VI. The Normal Distribution

VII. Investor Preferences under Uncertainty

VIII. Appendix A: Some Useful Probability Rules

IX. Additional Readings

Buzz Words: Probability Model, Value-at-Risk (VaR), Utility Theory
I. Readings and Suggested Practice Problems

BKM pp. 940-945, 948-969.
BKM Chapter 5: Section 5.2.
BKM Chapter 6.

Suggested Problems, Chapter 5: 4, 12-15, Chapter 6: 1, 6, 10.

II. Dealing with Uncertainty and Risk

Why do we need to measure uncertainty?

The returns on risky assets are random variables, so to manage an investment portfolio, we must know the “nature” of these random variables.

How to represent and measure uncertainty?

Example: A 3-State Probability Model

<table>
<thead>
<tr>
<th>State</th>
<th>Probability (Pr)</th>
<th>Ford</th>
<th>GM</th>
<th>Reebok</th>
<th>T-Bill</th>
</tr>
</thead>
<tbody>
<tr>
<td>BOOM (BM)</td>
<td>50%</td>
<td>25%</td>
<td>22%</td>
<td>9%</td>
<td>5%</td>
</tr>
<tr>
<td>NORMAL(NM)</td>
<td>30%</td>
<td>-3%</td>
<td>2%</td>
<td>9%</td>
<td>5%</td>
</tr>
<tr>
<td>RECESSION (RN)</td>
<td>20%</td>
<td>-10%</td>
<td>-19%</td>
<td>3%</td>
<td>5%</td>
</tr>
</tbody>
</table>
A. “**Probability**” vs. “**Statistics**”

Both probability and statistics deal with randomness / unpredictability / uncertainty / risk.

A *probability model* is an idealized conception of how reality works:
- “On the throw of a die, each of the six faces is equally likely to occur.”

*Statistics:* how do we use a *sample of data* to calibrate our probability model?
- “Given 1,000 tosses of the die, I will look at the proportion of times each face came up.”

B. **Constructing a probability model**

You can use *historical* return data to *approximate* the joint probability distribution of assets’ returns.

*For simplicity, abstract from the issue that this historical distribution is just an approximation of the true return distribution; instead, assume that this historical distribution is the true distribution.*

*(That is, assume that “statistics and probability coincide;” or, using statistical terminology, assume that sample-based estimates are the true population parameters.)*

From the study of Probability, we know that a distribution can be characterized via quantities called *moments*.

C. **Which moments to look at?**

1. The *first* moment of the return random variable is the *Expected Return* and it measures *Gains and Losses.*

2. *Second* moments of the return random variable, the *Variance* and *Covariances*, are measures of “*Risk.***

3. Can look at higher moments as well (*skewness, kurtosis*, etc.), but we focus on the first two …
**D. Meaning and Measurement of “Risk:” Suggestions**

1. An asset is risky if there is a possibility of any loss.

2. Risk is measured by Worst possible outcome.

3. **Value at Risk (VaR):**

   Value at risk is the measure suggested by the J.P. Morgan Riskmetrics system (http://www.jpmorgan.com). The VaR is a maximum expected dollar loss in a position (investment) over some predefined horizon. It is associated with a confidence level (e.g., what is $x$ such that we have a 90% chance of hitting our target plus-or-minus $x$?).

4. Some combined calculation that measures the extent of the loss and the likelihood of its occurrence.
   a. The lower semi-variance is: (E($r$) is defined on next page)
      \[
      \sum_s \Pr(s) \begin{cases} 
      [r(s) - E(r)]^2, & \text{if } r(s) < E(r) \\
      0, & \text{otherwise} 
      \end{cases}
      \]

      The semi-variance has difficult statistical and mathematical properties.
   
   b. Using **variance** (or **standard deviation**) to measure risk.
      
      (1) Easy to work with mathematically.
      (2) Measures overall uncertainty about negative and positive outcomes.
      (3) For a probability distribution that is symmetric about its mean (like the normal), high variance also means high lower-variance.
      (4) Does not reflect “skewness,” the asymmetry in the distribution.

5. **Covariation** with a portfolio or a macroeconomic variable.
III. How to calculate Expected Return

A. Formula

If there are $K$ possible states (the distribution is *discrete*): $s_1, s_2, \ldots, s_K$,

*Expected return on the asset:*

$$E[r_a] = \mu_a = \Pr(s_1) r_a(s_1) + \Pr(s_2) r_a(s_2) + \ldots + \Pr(s_K) r_a(s_K)$$

where $r_a(s)$ is the return on the asset in state $s$,

$\Pr(s)$ is the probability of state $s$.

B. Example

There are 3 states. So, using the above formula:

$$E[r_a] = \Pr(BM) r_a(BM) + \Pr(NM) r_a(NM) + \Pr(RN) r_a(RN)$$

Calculations:

$$E[r_{Ford}] = 0.50 \times 25\% + 0.30 \times -3\% + 0.20 \times -10\% = 9.6\%$$

$$E[r_{GM}] = 0.50 \times 22\% + 0.30 \times 2\% + 0.20 \times -19\% = 7.8\%$$

$$E[r_{Reebok}] = 0.50 \times 9\% + 0.30 \times 9\% + 0.20 \times 3\% = 7.8\%$$

$$E[r_{T-bill}] = 0.50 \times 5\% + 0.30 \times 5\% + 0.20 \times 5\% = 5\%$$
IV. How to calculate the Variance and Standard Deviation of Return

A. Formula

If there are $K$ possible states (the distribution is discrete): $s_1, s_2, \ldots, s_K$,

Variance of return on the asset:

$$\sigma^2[r_a] = \text{Var}[r_a] = E[(r_a - E[r_a])^2]$$

$$= \Pr(s_1)(r_a(s_1)-E[r_a])^2 + \Pr(s_2)(r_a(s_2)-E[r_a])^2 + \ldots + \Pr(s_K)(r_a(s_K)-E[r_a])^2$$

where $r_a(s)$ is the return on the asset in state $s$,
Pr(s) is the probability of state $s$.

Standard Deviation of return on the asset:

$$\sigma[r_a] = \sqrt{\sigma^2[r_a]}$$

B. Example

There are 3 states. So:

$$\sigma^2[r_a]=$$

$$=\Pr(BM)(r_a(BM)-E[r_a])^2 + \Pr(NM)(r_a(NM)-E[r_a])^2 + \Pr(RN)(r_a(RN)-E[r_a])^2$$

Calculations:

$$\sigma^2[r_{Ford}] = 0.50 \times (25-9.6)^2 + 0.30 \times (-3-9.6)^2 + 0.20 \times (-10-9.6)^2$$

$$= 118.58 + 47.628 + 76.832 = 243.04$$

$$\sigma[r_{Ford}] = \sqrt{243.04} = 15.5897\%$$
\[ \sigma^2[r_{GM}] = 0.50 \times (22-7.8)^2 + 0.30 \times (2-7.8)^2 + 0.20 \times (-19-7.8)^2 \\
= 100.82 + 10.092 + 143.648 = 254.56 \]
\[ \sigma[r_{GM}] = \sqrt{254.56} = 15.9549\% \]

\[ \sigma^2[r_{Reebok}] = 0.50 \times (9-7.8)^2 + 0.30 \times (9-7.8)^2 + 0.20 \times (3-7.8)^2 \\
= 0.72 + 0.432 + 4.608 = 5.76 \]
\[ \sigma[r_{Reebok}] = \sqrt{5.76} = 2.4\% \]

\[ \sigma^2[r_{T-bill}] = 0.50 \times (5-5)^2 + 0.30 \times (5-5)^2 + 0.20 \times (5-5)^2 \\
= 0 + 0 + 0 = 0 \]
\[ \sigma[r_{T-bill}] = \sqrt{0} = 0\% \]

### C. Some Highlights Using Real Monthly Data

<table>
<thead>
<tr>
<th>Asset</th>
<th>( \sigma[r] ) %</th>
</tr>
</thead>
<tbody>
<tr>
<td>IBM</td>
<td>8.004</td>
</tr>
<tr>
<td>Apple</td>
<td>13.050</td>
</tr>
<tr>
<td>Microsoft</td>
<td>8.203</td>
</tr>
<tr>
<td>Nike</td>
<td>9.265</td>
</tr>
<tr>
<td>ADM</td>
<td>6.712</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>2.886</td>
</tr>
<tr>
<td>Small Firm</td>
<td>3.711</td>
</tr>
<tr>
<td>Govt Bond</td>
<td>2.272</td>
</tr>
</tbody>
</table>

- High volatility (measured by std dev) of individual stocks
- Lower volatility of baskets of Securities
  (S&P 500, Small-Firm Portfolio, Government-Bond Portfolio)
- The Government-Bond Portfolio is as volatile as stocks over the short run
V. How to calculate the Covariance and Correlation between Two Returns

A. Formula

If there are $K$ possible states (the distribution is discrete): $s_1, s_2, \ldots, s_K$,

Covariance of the return on asset 1 with the return on asset 2:

\[
\sigma_{r_1, r_2} = \text{Cov}[r_{a1}, r_{a2}] = E[(r_{a1} - E[r_{a1}]) (r_{a2} - E[r_{a2}])]
\]

\[
= \Pr(s_1) (r_{a1}(s_1)-E[r_{a1}]) (r_{a2}(s_1)-E[r_{a2}])
+ \Pr(s_2) (r_{a1}(s_2)-E[r_{a1}]) (r_{a2}(s_2)-E[r_{a2}])
\ldots
+ \Pr(s_K) (r_{a1}(s_K)-E[r_{a1}]) (r_{a2}(s_K)-E[r_{a2}])
\]

where $r_{a1}(s)$ is the return on asset 1 in state $s$,
$r_{a2}(s)$ is the return on asset 2 in state $s$,
$\Pr(s)$ is the probability of state $s$.

Correlation of the return on asset 1 with the return on asset 2

\[
\rho_{r_1, r_2} = \frac{\sigma_{r_1, r_2}}{\sigma_{r_1} \sigma_{r_2}}
\]

B. Example

There are 3 states. So:

\[
\sigma_{r_1, r_2} = \Pr(BM) (r_{a1}(BM)-E[r_{a1}]) (r_{a2}(BM)-E[r_{a2}])
+ \Pr(NM) (r_{a1}(NM)-E[r_{a1}]) (r_{a2}(NM)-E[r_{a2}])
+ \Pr(RN) (r_{a1}(RN)-E[r_{a1}]) (r_{a2}(RN)-E[r_{a2}])
\]
Calculations:

\[ \sigma[r_{Ford}, r_{GM}] = 0.50 \times (25-9.6)(22-7.8) + 0.30 \times (-3-9.6)(2-7.8) + 0.20 \times (-10-9.6)(-19-7.8) \]
\[ = 109.34 + 21.924 + 105.056 = 236.32. \]
\[ \rho[r_{Ford}, r_{GM}] = 236.32/(15.5897 \times 15.9549) = 0.95009. \]

\[ \sigma[r_{Ford}, r_{Reebok}] = 0.50 \times (25-9.6)(9-7.8) + 0.30 \times (-3-9.6)(9-7.8) + 0.20 \times (-10-9.6)(3-7.8) \]
\[ = 9.24 - 4.536 + 18.816 = 23.52. \]
\[ \rho[r_{Ford}, r_{Reebok}] = 23.52/(15.5897 \times 2.4) = 0.6286. \]

\[ \sigma[r_{GM}, r_{Reebok}] = 0.50 \times (22-7.8)(9-7.8) + 0.30 \times (2-7.8)(9-7.8) + 0.20 \times (-19-7.8)(3-7.8) \]
\[ = 8.52 - 2.088 + 128.64 = 32.16. \]
\[ \rho[r_{GM}, r_{Reebok}] = 32.16/(15.9549 \times 2.4) = 0.8398. \]

\[ \sigma[r_{Ford}, r_{T-bill}] = 0.50 \times (25-9.6)(5-5) + 0.30 \times (-3-9.6)(5-5) + 0.20 \times (-10-9.6)(5-5) \]
\[ = 48 - 48 = 0. \]
### C. Some Highlights Using Real Monthly Data

<table>
<thead>
<tr>
<th>$\rho_{r_i,r_j}$</th>
<th>$i =$</th>
<th>IBM</th>
<th>Apple</th>
<th>Microsoft</th>
<th>Nike</th>
<th>ADM</th>
<th>S&amp;P 500</th>
<th>Small Firm</th>
<th>Govt Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j =$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IBM</td>
<td>1.000</td>
<td>0.226</td>
<td>0.320</td>
<td>0.145</td>
<td>-0.093</td>
<td>0.228</td>
<td>0.045</td>
<td>-0.104</td>
<td></td>
</tr>
<tr>
<td>Apple</td>
<td>0.226</td>
<td>1.000</td>
<td>0.441</td>
<td>0.321</td>
<td>0.243</td>
<td>0.360</td>
<td>0.473</td>
<td>-0.026</td>
<td></td>
</tr>
<tr>
<td>Microsoft</td>
<td>0.320</td>
<td>0.441</td>
<td>1.000</td>
<td>0.312</td>
<td>0.161</td>
<td>0.372</td>
<td>0.395</td>
<td>-0.007</td>
<td></td>
</tr>
<tr>
<td>Nike</td>
<td>0.145</td>
<td>0.321</td>
<td>0.312</td>
<td>1.000</td>
<td>0.281</td>
<td>0.372</td>
<td>0.260</td>
<td>0.158</td>
<td></td>
</tr>
<tr>
<td>ADM</td>
<td>-0.093</td>
<td>0.243</td>
<td>0.161</td>
<td>0.281</td>
<td>1.000</td>
<td>0.425</td>
<td>0.182</td>
<td>0.379</td>
<td></td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>0.228</td>
<td>0.360</td>
<td>0.372</td>
<td>0.372</td>
<td>0.425</td>
<td>1.000</td>
<td>0.621</td>
<td>0.480</td>
<td></td>
</tr>
<tr>
<td>Small Firm</td>
<td>0.045</td>
<td>0.473</td>
<td>0.395</td>
<td>0.260</td>
<td>0.182</td>
<td>0.621</td>
<td>1.000</td>
<td>0.098</td>
<td></td>
</tr>
<tr>
<td>Govt Bond</td>
<td>-0.104</td>
<td>-0.026</td>
<td>-0.007</td>
<td>0.158</td>
<td>0.379</td>
<td>0.480</td>
<td>0.098</td>
<td>1.000</td>
<td></td>
</tr>
</tbody>
</table>

**Note:**
- Correlation of Return(asset1, asset1) = 1
- IBM is more correlated with high tech stocks than with sportswear or food stocks. It is also more correlated with large-stock portfolio (S&P 500) than with a Small Firm portfolio.  
  *This all makes sense, and shows that correlation (or covariance) is a useful and meaningful summary statistic for real world purposes, and for characterizing return distributions.*
- ADM (food industry) is negatively correlated with IBM (tech industry). 
  *How can we use this information for portfolio selection?*
- S&P 500 and Government Bonds are highly positively correlated (Stocks and Bonds move together more often than they move in opposite directions!) *Why?*
VI. The Normal Distribution

A. The standard normal distribution

A standard normal random variable has a mean ($\mu$) of zero and a standard deviation ($\sigma$) of one, and a continuum of possible values.

- Suppose $r \sim N(0,1)$ is a daily return on an asset.
- Area lying to the left of (-2) = $Pr[r \leq -2] = (1-0.954)/2 = 0.023$
- Let $N(d)$ denote the cumulative distribution function of a standard normal random variable: $N(d) = Pr[r \leq d]$ (i.e., the area of the “bell” lying to the left of $d$), then
  $N(-2) = Pr[r \leq -2] = 0.023$
  $N(2) = Pr[r \leq 2] = 1 - Pr[r > 2] = 1 - Pr[r \leq -2] = 1 - N(-2) = 0.977$
- [The “bell” area between (-1) and 1] = $Pr[-1 < r \leq 1] = N(1) - N(-1) = 0.683$
  So, 68.3% confidence interval is (-1,1).
  This means that “on 683 days out of 1000 have: -1 < r \leq 1 ”
- 90% confidence interval is (-1.64, 1.64)
**B. The “Z” Score**

The “z” score for a normal distribution is the number of standard deviations plus-or-minus (“±”) away from the mean for a given “confidence region.”

- Suppose \( r \sim N(\mu, \sigma) \).
- Then, \( y = (r - \mu) / \sigma \sim N(0, 1) \)
- A 90% confidence interval for \( y \) is (-1.64, 1.64),
  
i.e., \( 0.9 = \Pr[-1.64 < y \leq 1.64] \)

A 90% confidence interval for \( r \) is \( (\mu - 1.64 \sigma, \mu + 1.64 \sigma) \),
  
i.e., \( 0.9 = \Pr[\mu - 1.64 \sigma < r \leq \mu + 1.64 \sigma] \)
- \( \Pr[r \leq \mu - 1.64 \sigma] = \Pr[(r - \mu) / \sigma \leq -1.64] = \Pr[y \leq -1.64] = (1-0.9)/2 = 0.05 \)

So, the probability that \( r \) is 1.64 standard deviations, or more, below its mean \( \mu \), equals the probability of a standard normal random variable to be below \(-1.64\), which corresponds to the 0.05 area in the lower tail of the distribution.
C. Example: Losses on investments in large stocks

Annual large company stock returns have (using 1926-1996 data) mean annual return = 12.5%, sample standard deviation = 20.4%. The minimum return was -46%. What is the probability of observing a return of -46% or lower in any given year?

Answer

Assume the above sample moments are the true moments of a normal distribution.

Then,

\[
Pr[r \leq -46\%] = Pr\left(\frac{r - 12.5}{20.4} \leq \frac{-46 - 12.5}{20.4}\right) = Pr\left(\frac{r - 12.5}{20.4} \leq -2.9\right) = Pr[y \leq -2.9] = N(-2.9) = 1 - N(2.9) = 0.0019.
\]

Alternatively, note that: \((-46-12.5)/20.4=-2.9\), (i.e., \(z = 2.9\)), so that -46% is 2.9 standard deviations below the mean.

Need to use a table (or software) for a normal distribution to find the area lying to the left of -2.9 (that is \(N(-2.9)\)).

Or, equivalently, because the normal probability density is symmetric, to find the area lying to the right of 2.9 (that is \(1-N(2.9)\)).

This area equals 0.0019, or 0.19%.

(We interpret this as observing return of -46%, or lower, about 2 years out of 1000 years.)
D. Example: VaR

A return is normally distributed with $\mu = 10\%$ and $\sigma = 5\%$. You have $10,000$ invested in this asset. What is the dollar amount $x$ such that 90% of the time you will have what you expect $\pm x$?

**Answer**

Let $r$ be the rate of return on the investment, and $(1+r)$ the total return: $r \sim N(0.1, 0.05)$, i.e., $r$ is normal with $E[r] = 0.1$, $\sigma[r] = 0.05$.

$\$ rate of return: $10,000r$

$$E[10,000r] = 10,000 E[r] = 10,000 \times 0.1 = 1,000$$

$$\sigma[10,000r] = 10,000 \sigma[r] = 10,000 \times 0.05 = 500$$

$\$ total return: $10,000(1+r) = 10,000 + 10,000r$

$$E[10,000(1+r)] = 10,000 + E[10,000r] = 10,000 + 1,000 = 11,000$$

$$\sigma[10,000(1+r)] = \sigma[10,000 + 10,000r] = \sigma[10,000r] = 500$$

Also note that because $\$ total return is a linear combination of a normal random variable $(r)$, it is also a normal random variable. Therefore: $10,000(1+r) \sim N(11,000, 500)$.

Now can use the Z score; we know that for a 90% confidence interval: 0.9 = $\Pr[11,000 - 1.64 \times 500 < 10,000(1+r) \leq 11,000 + 1.64 \times 500]$]

We want to find $x$ such that:

0.9 = $\Pr[11,000 - x < 10,000(1+r) \leq 11,000 + x]$

So, $x = 1.64 \times 500 = 820$.

Only with probability of 5% or less you will have less than $10,180. Hence, at a 5% confidence level your Value-at-Risk (VaR), relative to your expected gain, is $820. (Relative to your initial position you don’t lose, at this confidence level, but rather gain $180.)
VII. Investor Preferences under Uncertainty

A. Summarizing Tastes and Preferences

Assume a one period setting (e.g., “today” and “tomorrow”).

Note that a normal return distribution is completely characterized by:

- Expected Return over the Period, $E[r]$.
- Standard Deviation of Return over the Period, $\sigma[r]$.

This is one sufficient condition for individuals to only care about their one-period expected portfolio return and about their portfolio’s standard deviation.

B. Risk Aversion

One of the cornerstones of modern finance is that individuals are risk averse (and prefer more to less)

For a risk averse individual, the following is “reasonable”

For a given expected portfolio return, prefer a portfolio with a lower standard deviation of return.

For a given standard deviation of portfolio return, prefer a portfolio with a higher expected return.

(We will assume the above is “true”, but how reasonable is this?)
C. Mean-Variance Portfolio Analysis

Assume investors base their decisions on:
the mean (expected return) of a portfolio,
and the return variance (or standard deviation).

Suppose that you are comparing two investments under this criterion and you must pick one or the other.

- If $E[r_1] > E[r_2]$ and $\sigma_1 < \sigma_2$, then 1 is preferable to 2: 
  *(This does not mean that 1 will outperform 2 in all circumstances.)*

- If $E[r_1] > E[r_2]$ and $\sigma_1 > \sigma_2$, then we can’t in general say which one is better.
  In such cases, we must appeal to utility theory.

D. Utility Theory

Assume that investor’s tastes and preferences can be represented by indifference curves on a ”risk/return” graph:

At all points on an indifference curve, the investor enjoys the same level of utility.
In $\{\text{Standard Deviation of Return, Expected Return}\}$ space, a risk averse individual’s indifference curves have positive slopes: Since a risk averse individual likes mean but dislikes standard deviation, the only way the individual can accept more standard deviation and maintain the same level of utility is if she is given a higher expected return.

For any individual, as you move north in $\{\sigma[r], E[r]\}$ space, utility is increasing.

For any individual, her indifference curves can not cross since that would imply that a particular $\{\sigma[r], E[r]\}$ combination was associated with two levels of utility.

However, the trade-off between risk and return for any two risk averse individuals may be completely different: see individuals A and B above. Individual B is more risk averse than A since at any point in $\{\sigma[r], E[r]\}$ space, B’s indifference curve has a steeper slope.
VIII. *Appendix A*: Some Useful Probability Rules

**A. Expected Return**
many forms of notation can be used: $E[r]$, $E(r)$, $Er$, …

$E[a + r] = a + E[r]$ , where $a$ is a constant
$E[a r] = a E[r]$
$E[r_1 + r_2] = E[r_1] + E[r_2]$

**B. Variance and Standard Deviation**
many forms of notation can be used: $\sigma^2[r]$, $\sigma^2(r)$, $\sigma_r^2$, … $\sigma[r]$, $\sigma_r$, …

$\sigma^2[r] = E[(r - E[r])^2] = E[r^2] - (E[r])^2$

$\sigma^2[a + r] = \sigma^2[r]$

$\sigma^2[a r] = a^2 \sigma^2[r]$

$\sigma[a + r] = \sigma[r]$

$\sigma[a r] = |a| \sigma[r]$

**C. Covariance**
many forms of notation can be used: $\sigma[r_1, r_2]$, $\sigma^2(r_1, r_2)$, $\sigma^2_{r_1,r_2}$, $\sigma^2_{1,2}$, …

$\sigma[r_1 , r_2] = E[(r_1 - E[r_1])(r_2 - E[r_2])] = E[r_1 r_2] - E[r_1] E[r_2]$

$\sigma[a , r] = 0$

$\sigma[a_1 r_1 , a_2 r_2] = a_1 a_2 \sigma[r_1 , r_2]$

$\sigma[r_1 , r_2] = \sigma[r_2 , r_1]$

$\sigma[r_1 + r_2 , r_3] = \sigma[r_1 , r_3] + \sigma[r_2 , r_3]$

$\sigma^2[a_1 r_1 + a_2 r_2] = a_1^2 \sigma^2[r_1] + a_2^2 \sigma^2[r_2] + 2 a_1 a_2 \sigma[r_1 , r_2]$

IX. Additional Readings