Lecture Notes 6

Asset Allocation: Risky vs. Riskless

I. Readings and Suggested Practice Problems

II. Expected Portfolio Return: General Formula

III. Standard Deviation of Portfolio Return: One Risky Asset and a Riskless Asset

IV. The Asset Allocation Framework

V. The Capital Allocation Line

VI. Portfolio Management: One Risky Asset and a Riskless Asset

VII. Additional Readings

Buzz Words: Reward to Variability Ratio, Separation Theorem, Portfolio’s Risk Adjustment, Portfolio Management
I. Readings and Suggested Practice Problems

BKM, Chapter 7  
*Suggested* Problems, Chapter 7: 8, 18-23.

A useful example from the Additional Readings:

**Investment Advice from Wall Street**

Two annual recommendations from Morgan Stanley (as reported in the WSJ)

<table>
<thead>
<tr>
<th></th>
<th>Stocks</th>
<th>Bonds</th>
<th>Cash</th>
</tr>
</thead>
<tbody>
<tr>
<td>% in Complete Port.</td>
<td>% in Complete Port.</td>
<td>% in Comp</td>
<td></td>
</tr>
<tr>
<td>[% in the risky part]</td>
<td>[% in the risky part]</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Year 1</strong></td>
<td>65</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>[81.25]</td>
<td>[18.75]</td>
<td></td>
</tr>
<tr>
<td><strong>Year 2</strong></td>
<td>80</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>[80]</td>
<td>[20]</td>
<td></td>
</tr>
</tbody>
</table>

In this and subsequent lectures, we want to understand:

- How to chose the relative weights, in *brackets* above, of “Stocks” and “Bonds” (or any other two *risky* assets that make up the risky portion of your complete portfolio).

- What is the role of “Cash” (i.e., the riskless asset within the portfolio, such as a money market account).
II. Expected Portfolio Return: General Formula

Recall the portfolio-return formula (from Lecture Notes 4b), and apply to it the expectation operator (using the rules in Appendix A of the previous Lecture Notes) to get:

\[
E[r_p(t)] = w_{1,p} E[r_1(t)] + w_{2,p} E[r_2(t)] + \ldots + w_{N,p} E[r_N(t)]
\]

where

- \(N\) is the number of assets in the portfolio,
- \(E[r_i(t)]\) is the expected return on asset \(i\) in period \(t\),
- \(w_{i,p}\) is the weight of asset \(i\) in the portfolio \(p\) at the start of period \(t\),
- \(E[r_p(t)]\) is the expected return on portfolio \(p\) in period \(t\).

The above formula holds for any number of assets and with or without the riskless asset as one of the assets in the portfolio.

Example

(continued from previous Lecture Notes): Consider a portfolio with 80% invested in Ford and the remaining 20% invested in T-bills. The expected return on this portfolio, given the expected returns on the stocks of Ford and T-bills (which we calculated in the previous lecture) is therefore:

\[
E[r_p] = w_{\text{Ford},p} E[r_{\text{Ford}}] + w_{\text{T-bill},p} E[r_{\text{T-bill}}]
\]

\[
= 0.8 \times 9.6\% + 0.2 \times 5\%
\]

\[
= 8.68\%
\]
III. Standard Deviation of Portfolio Return: One Risky Asset and a Riskless Asset

When the portfolio is composed from two assets ($N=2$), where one is a risky asset, labeled “asset $i$,” and the second is riskless, then the standard deviation of the portfolio (applying the variance rule to the portfolio return formula, $r_p(t) = w_{i,p} r_i(t) + (1-w_{i,p}) r_f(t)$, and taking the square root) is given by:

$$
\sigma[r_p(t)] = |w_{i,p}| \sigma[r_i(t)]
$$

where

- $\sigma[r_i(t)]$ is the standard deviation of the return on risky asset $i$ in period $t$,
- $|w_{i,p}|$ is the absolute value of the weight of asset $i$ in the portfolio $p$,
- $\sigma[r_p(t)]$ is the standard deviation of the return on the portfolio $p$ in period $t$.

The above formula holds only when one asset is risky and the other is riskless.

**Example**

Consider the portfolio with 80% invested in Ford (the risky asset) and the remaining 20% invested in T-bills (the riskless asset):

$$
\sigma[r_p] = |w_{Ford,p}| \sigma[r_{Ford}]
= 0.8 \times 15.5897\% = 12.4718\%.
$$
IV. The Asset Allocation Framework

*Asset Allocation:* The allocation of funds among *broad* asset classes such as domestic stocks, foreign stocks, bonds, and cash.

**A. Top-down asset allocation**

1. Assume you are given a portfolio of *risky* securities; Call this portfolio $P$.

   It can be *any* portfolio familiar to you (e.g., the S&P 500, the small-firm portfolio) or some “best” mix of risky securities or portfolios (e.g., as recommended by your broker [see the WSJ articles in the additional readings, where $P$ is a combination of “Stocks” and “Bonds”]).

2. You will decide how much to put in $P$, and how much in a risk-free asset (e.g., cash, T-bill, money market fund).

   As usual, we use the notation:

   $r_P$ is the (random) return on the risky-portfolio $P$
   $r_f$ is the return on the risk-free asset

   The *complete*, or *combination*, or *combined* portfolio, (composed from $P$ and a riskless asset) is denoted $C$, and is described by the following portfolio weights:

   $y$ is the fraction of $C$ invested in the portfolio $P$ (i.e., $w_{P,C} = y$)
   $(1-y)$ is the fraction in a risk-free security (i.e., $w_{f,C} = 1-y$)
B. Example for how a Combined Portfolio may look like

You have $10,000 to invest. Following your broker’s recommendation you have the risky asset $P$ as 60% stock and as 40% bonds (recall that even government bonds are risky, unless held to maturity).

When you choose $y = 0.6$, the combined portfolio $C$ is:

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stocks</td>
<td>3,600</td>
</tr>
<tr>
<td>Bonds</td>
<td>2,400</td>
</tr>
<tr>
<td>Total in $P$</td>
<td>6,000</td>
</tr>
<tr>
<td>Risk-free</td>
<td>4,000</td>
</tr>
<tr>
<td>Net Worth</td>
<td>10,000</td>
</tr>
<tr>
<td>Total Assets</td>
<td>10,000</td>
</tr>
</tbody>
</table>

With $y = 0.8$, the combined portfolio $C$ is:

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stocks</td>
<td>4,800</td>
</tr>
<tr>
<td>Bonds</td>
<td>3,200</td>
</tr>
<tr>
<td>Total in $P$</td>
<td>8,000</td>
</tr>
<tr>
<td>Risk-free</td>
<td>2,000</td>
</tr>
<tr>
<td>Net Worth</td>
<td>10,000</td>
</tr>
<tr>
<td>Total Assets</td>
<td>10,000</td>
</tr>
</tbody>
</table>
C. Describing the Properties of the Combined portfolio \( C \)

Using our knowledge of the return formula and its properties for the case of one risky and a riskless asset, we know that:

1. In any given period, the \textit{return} on \( C \) is:

\[
r_C = y r_P + (1-y) r_f = r_f + y (r_P - r_f)
\]

[e.g., \( r_P = 20\% \), \( r_f = 5\% \), \( y = 0.6 \) \( \Rightarrow \) \( r_C = 14\% \)]

2. The \textit{expected return} on \( C \) is:

\[
E[r_C] = r_f + y (E[r_P] - r_f)
\]

3. The \textit{standard deviation} of the return of \( C \) is (assume \( y \geq 0 \)):

\[
\sigma_C = y \sigma_P
\]

where \( \sigma_P \) is the standard deviation of the return on \( P \).

[e.g., using approximate numbers (CRSP-SBBI):
If \( P \) is an S&P Composite Index portfolio, then
\[E[r_P] - r_f = 12.5\% - 3.8\% = 8.7\% \text{ per year}; \sigma_P = 20.4\% \text{ per year.} \]

4. Substituting \( y = \sigma_C / \sigma_P \) in \( E[r_C] = r_f + y (E[r_P] - r_f) \) yields the

\[
\text{Capital Allocation Line (CAL): } E[r_C] = r_f + \left\{ \frac{(E[r_P] - r_f)}{\sigma_P} \right\} \sigma_C
\]
V. The Capital Allocation Line

A. The CAL gives the trade-off between risk and return

That is, the CAL describes all risk-return combinations available to investors.

It allows to see what expected return on the combined portfolio (E[r_C]) is attainable for a given level of risk (σ_C).

\[
\frac{E_{r_p} - r_f}{\sigma_p} \quad \text{is the Reward-to-Variability ratio of } P
\]

\[
E_{r_p} - r_f \quad \text{is the Risk Premium on } P
\]
B. The exact location of the combined portfolio C on the CAL depends on the investor’s risk-return preferences.

Example

Assume that you know that $E[r_P] - r_f = 9\%, r_f = 5\%, \sigma_P = 20\%$, and you choose $y = 0.6$, (y is your personal choice, reflecting your preferences for risk and reward).

Then, $\sigma_C = 0.6 \times 20\% = 12\%$.

$Er_C \equiv E[r_C] = 5\% + (9\% / 20\%) \times 12\% = 10.4\%$. 

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C. Buying on Margin and Selling Short

1. Buying $P$ on margin, allows you to construct a portfolio $C$ to the right of $P$ on the CAL.

2. Short-Selling $P$ (i.e., when $y < 0$), allows you to construct a portfolio $C$ on the “mirror-image of the CAL” (where $\sigma_C = -y \sigma_P$), so that:

\[
E[r_C] = r_f - \frac{(E[r_P]-r_f)}{\sigma_P} \sigma_C
\]

3. What happens when $E[r_P] < r_f$?
VI. Portfolio Management:  
One Risky Asset and a Riskless Asset

A. How would a risk averse investor choose the portfolio weights for a portfolio consisting solely of the riskless asset and a given risky asset?

1. For any point on the negative sloped part, a risk averse individual, with mean-variance preferences, is going to prefer at least one point on the positive sloped part of the curve (the one with the same standard deviation and a higher expected return).

2. If the expected return on a risky asset exceeds the riskless rate \( \mathbb{E}[r_P] > r_f \): an individual forming a portfolio using only that asset and the riskless asset will not want to short sell the risky asset (not want \( y < 0 \)), but the individual may want to buy it on margin (may want \( y > 1 \)).

3. If the expected return on a risky asset is less than the riskless rate \( \mathbb{E}[r_P] < r_f \): an individual forming a portfolio using only that asset and the riskless asset will want to short sell the risky asset (will want \( y < 0 \)).

4. The exact weight that the individual wants to hold of the risky asset depends on her attitudes to risk; different individuals will choose to hold different amounts of the risky asset.

Who wants to hold positive amounts of both the risky asset and T-bills \( (0<y<1) \)?
Who wants to buy the risky asset on margin \( (y>1) \)?
**B. Separation Property**

The portfolio of risky assets, $P$, is selected in a separate step, independent of investors’ attitude for risk. Any (mean-variance) investor, regardless of risk-taking preferences will prefer the portfolio $P$ with the highest CAL.

**Implications of the Separation Property for Portfolio Management**

1. Investment advisers can recommend a single optimal portfolio, with the highest CAL

   That’s what they are actually doing in practice, except that different investment advisers may disagree on the portfolio because they differ in their opinion about the first and second moments of the risky assets.

2. Investors can use the riskfree asset to adjust investment risk.

   The “Cash” in Morgan Stanley’s recommendation is thus used for risk adjustment of the portfolio.

**C. The Portfolio Choice Process: Summary so far**

So far, we saw two steps to construct your optimal investment portfolio -- assuming that mean-variance preferences are a good description of your risk profile (e.g., if you are risk averse and returns are normally distributed):
Stage I: Asset Selection
Determine the “best” $P$ available to you in the mean/standard-deviation space.
(For that you need to use the characterization of the return distribution. So far, we took $P$ as given. The “identity” of $P$ is the subject of the subsequent lectures on asset selection and equilibrium; this will allow us to better understand how, for example, Morgan Stanley chose the weights of stocks and bonds within $P$).

Use $P$ to determine the “best” CAL (In our setting it is the same for all investors that have the same $P$; in other words, Asset Selection is an objective procedure).

Stage II: Asset Allocation
Find the combined portfolio $C$ that “fits your personal risk profile” as the point on the CAL providing you with the “highest utility.” ($C$ is likely to differ across investors, even if they have the same $P$; Asset Allocation is subjective).

Note: In the next lecture we will see how determining the “best $P$” and the “best” CAL is done simultaneously.

VII. Additional Readings

The following articles present the evolution of quarterly Asset-Allocation of Wall Street strategists over time, which are the source for our example.

In the context of our discussion, you should view “Stocks” and “Bonds” as two components of the risky portfolio $P$, and compare the weight of $P$ to that of “Cash” (the riskless asset).
[As noted, we took the composition of $P$ as given, but we will discuss how to construct $P$ in the next lecture. Therefore, you may want to take a look at these articles again later].