APPENDIX

Proof

The payoff to the holder of a non-dividend paying stock plus a European-style contingent payment put option (struck at-the-money) is:

\[
S_T \begin{cases} 
S_T \geq S \\ S_T < S
\end{cases} + S - V \begin{cases} 
S_T \geq S \\ S_T < S
\end{cases} \quad (A1)
\]

where \( S_T \) is the terminal price of the stock, \( S \) is the current price of the stock, and \( V \) is the contingent payment.

To solve for \( V \), assume that \( S_T \) is lognormally distributed, let \( \mu \) represent the expected return of \( S \), and let \( \eta(z) \) represent the standard normal density function. That is,

\[
\eta(z) = e^{-\frac{(z^2/2)}{\sqrt{2\pi}}}. \quad (A2)
\]

Consider the first term in equation (A1). Under the assumption that \( S_T \) is lognormal, we have:

\[
E \left( S_T \bigg| S_T \geq S \right) = \int_{S_T \geq S} S_T L(S_T) \, dS_T \quad (A3a)
\]

\[
= \int_{-\mu T/\sigma \sqrt{T}}^{+\infty} S e^{\mu T + \sigma \sqrt{T}z} e^{-\frac{(z^2/2)}{\sqrt{2\pi}}}(dz) \quad (A3b)
\]

\[
= S e^{\mu T} \int_{-\mu T/\sigma \sqrt{T}}^{+\infty} e^{(\sigma \sqrt{T})z} - (\frac{z^2}{2})/(\sqrt{2\pi}) \, (dz) \quad (A3c)
\]

\[
= S e^{\mu T} \int_{-\mu T/\sigma \sqrt{T}}^{+\infty} e^{(\sigma^2 \sqrt{T}^2/2)} - (\frac{z^2}{2})/(\sqrt{2\pi}) \, (dz) \quad (A3d)
\]

\[
= S e^{\mu T + (\sigma^2 \sqrt{T}^2/2)} \int_{-\mu T/\sigma \sqrt{T}}^{(\mu T/\sigma \sqrt{T}) + \sigma \sqrt{T}} e^{-(y^2/2)/(\sqrt{2\pi})} \, (dy) \quad (A3e)
\]

\[
= S e^{\mu T + (\sigma^2 \sqrt{T}^2/2)} N(d_1), \quad (A3f)
\]

where \( d_1 = (\mu T/\sigma \sqrt{T}) + \sigma \sqrt{T} \).
In line (A3a) the conditional expected value of $S_T$ is expressed in integral form where \( L(S_T) \) is the lognormal density function for $S_T$. In line (A3b) a change in variables is performed on $S_T$, and the density function of the standardized normal variable, $z$, is written out. In line (A3c), $Se^{\mu T}$ is factored out of the integral and the square in the exponent within the integral is completed. In (A3d), $e^{(\sigma^2 T)/2}$ is factored out of the integral and the remaining expression in the exponent within the integral is simplified. In (A3e), a change in variables $y = \sigma \sqrt{T} - z$ is performed and the limits of the integration are redefined. Finally, line (A3f) simplifies the expression.

For a risk-neutral economy, $\mu = b - \sigma^2/2$, where $b$ is the asset’s cost of carry. For a non-dividend paying stock, $b = r$, so $\mu = r - \sigma^2/2$. Thus from line (A3f),

$$Se^{[r - (\sigma^2/2)] T + (\sigma^2 T)/2} N(d_1) = Se^{rT} N(d_1). \quad (A4)$$

Equation (A4) completes the first part of the payoff.

Solving the second part of the payoff, $S - V \bigg|_{S_T < S}$, is relatively easy:

$$S - V \bigg|_{S_T < S} = (S - V) \text{Prob}(S_T < X). \quad (A5)$$

First compute $\text{Prob}(S_T \geq X)$:

$$\text{Prob} (S_T \geq X) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-((\mu T)/\sigma\sqrt{T})} dz = \int_{-\infty}^{+\infty} \eta(z)dz = N(d_2), \quad (A6)$$

where $d_2 = \mu T/\sigma \sqrt{T}$.

From the symmetry of the standard normal density function, $\text{Prob}(S_T < S) = 1 - N(d_2) = N(-d_2)$. Thus we have:

$$S - V \bigg|_{S_T < S} = (S - V) N(-d_2). \quad (A7)$$

Equation (A7) completes the second part of the payoff.

Now combine equations (A4) and (A7) to obtain the expression for the total payoff:

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\(^1\) Where $y = -z$, \[ \int_{-\infty}^{d_1} e^{-((\sigma^2/2) (\sqrt{2\pi}))} (dz) = \int_{-\infty}^{d_1} e^{-((y^2/2) (\sqrt{2\pi}))} (dy). \]
\[ Se^{rT} N(d_1) + (S - V) N(-d_2). \]  

(A8)

By the law of one price, if the client pays nothing today for the contingent payment put, then the present value of the total payoff (A8) must be equal to the current value of the underlying asset (the investment required to buy the stock). Put another way, the total investment today is \( S \). Therefore, in a risk-neutral economy,

\[ e^{-rT} [Se^{rT} N(d_1) + (S - V) N(-d_2)] = S, \]  

(A9a)

or \[ S N(d_1) + (S - V)e^{rT} N(-d_2) = S, \]  

(A9b)

\[ N(d_1) + (S - V)/(S)e^{rT} N(-d_2) = 1, \]  

(A9c)

\[ (S-V)/(S)e^{rT} = [1- N(d_1)]/N(-d_2) = N(-d_1)/N(-d_2), \]  

(A9d)

\[ (S-V)/S = N(-d_1)e^{rT}/N(-d_2) \]  

(A9e)

\[ S - V = [N(-d_1)e^{rT}/N(-d_2)]S, \]  

(A9f)

\[ V = S - [N(-d_1)e^{rT}/N(-d_2)]S, \]  

(A9g)

\[ V = S \{1 - [N(-d_1)e^{rT}/N(-d_2)]\} \]  

(A9h)

where \( d_1 = [r + (\sigma^2/2)T]/\sigma\sqrt{T} \), and \( d_2 = d_1 - \sigma\sqrt{T} \).

Equation (A9h) is the equation for quantifying the contingent payment, \( V \).

\[ 2 \text{ It is interesting to note that as } V = P e^{rT}/N(-d_2), \text{ where } P \text{ is the price of a plain-vanilla put option, substituting (A9h) for } V \text{ in this expression and setting } S = X \text{ leads to the original Black and Scholes (1973) model for pricing plain vanilla put options:} \]

\[ S \{1 - [N(-d_1)e^{rT}/N(-d_2)]\} = Pe^{rT}/N(-d_2), \]

\[ \bullet \]

\[ \bullet \]

\[ \bullet \]

\[ P = Xe^{rT}N(-d_2) - SN(-d_1). \]