The midterm will last 90 minutes and will have three questions, each of equal marks. Within each question there will be a number of parts (say 4-6 parts per question) and the weight given to each part will also be indicated.

Here are two sample questions.

**Question 1. Solow Growth Model (30 marks):** Let time be discrete, \( t = 0, 1, \ldots \). Let the national resource constraint be

\[
C_t + I_t = Y_t = F(K_t, A_t, N_t)
\]

where \( C_t \) denotes aggregate consumption, \( I_t \) denotes aggregate investment, \( Y_t \) denotes aggregate output, \( K_t \) denotes aggregate capital, and \( N_t \) denotes the level of the working age population. Labor-augmenting technological progress is denoted \( A_t \). The production function is constant returns in capital and labor with always positive but diminishing marginal product. In particular,

\[
\lim_{K \to 0} F_K(K, AN) = \infty
\]

\[
\lim_{K \to \infty} F_K(K, AN) = 0
\]

Both labor and technology grow exogenously

\[
N_{t+1} - N_t = nN_t, \quad n > 0, \quad N_0 \text{ given}
\]

\[
A_{t+1} - A_t = gA_t, \quad g > 0, \quad A_0 \text{ given}
\]

Let capital accumulation be given by

\[
K_{t+1} = (1 - \delta)K_t + I_t, \quad 0 < \delta < 1, \quad K_0 \text{ given}
\]

where \( \delta \) is the physical depreciation rate. Finally, let consumption be a fixed fraction of national output

\[
C_t = (1 - s)Y_t, \quad 0 < s < 1
\]

where \( s \) denotes the national saving rate.
(a) (5 marks): Show that this model can be reduced to a single non-linear difference equation in a state variable $x_t$, i.e., that you can write the model as

$$x_{t+1} = \psi(x_t), \quad x_0 \text{ given}$$

What is $x_t$? Provide an explicit formula for the function $\psi$.

(b) (5 marks): Let $\bar{x}$ denote a fixed point of $\psi$. How many fixed points does $\psi$ have? Linearize the function $\psi$ around each of its fixed points and determine the local stability or instability of each such point.

(c) (5 marks): Suppose that the aggregate production function is Cobb-Douglas with capital share $\alpha$,

$$F(K, AN) \equiv K^\alpha(AN)^{1-\alpha}, \quad 0 < \alpha < 1$$

Provide an explicit solution for each of the function $\psi$’s fixed points. Explain how the fixed points depend on the economic parameters of the model (especially $\alpha, s, \delta, g, n$). Give economic intuition.

(d) (5 marks): Use a log-linearization argument to determine the approximate speed of convergence to each steady state. Explain how the speed of convergence depends on the economic parameters of the model.

(e) (10 marks): Suppose that the production function is hit by random shocks

$$F(K_t, A_tN_t) \equiv z_t K_t^\alpha(A_tN_t)^{1-\alpha}$$

where the $\log(z_t)$ are IID Gaussian with mean 0 and variance $\sigma^2$. Use a log-linearization to derive an approximate linear stochastic difference equation in the state $\hat{x}_t$ and the shocks $\hat{z}_t$. Solve for the stationary distribution of $\hat{x}_t$ and explain how its mean and variance depend on the parameters of the model.
Question 2. Stochastic Labor Supply (30 marks): Let time be discrete, \( t = 0, 1, \ldots \). Suppose that a worker faces a stochastic real wage rate each period which follows an \( n \)-state Markov chain \((w, P, \pi_0)\) where \( w \) is an \( n \)-vector, \( P \) is a transition matrix and \( \pi_0 \) is an initial distribution. Suppose that each period, the worker solves the static utility maximization problem over consumption \( c \) and leisure \( \ell \)

\[
\max_{c_t, \ell_t} U(c_t, \ell_t)
\]

subject to a budget constraint and a constraint on the time endowment

\[
c_t = w_t n_t \\
n_t + \ell_t = 1
\]

where \( w_t \) is this period’s random wage realization and \( n_t \) is the fraction of time allocated to working. Suppose that \( U(c, \ell) \) is strictly increasing and strictly concave in each argument.

(a) (5 marks): Explain how a labor supply schedule of the form

\[
n = \varphi(w)
\]

can be derived from the optimization problem. Explain how you characterize the function \( \varphi \). What assumption do preferences have to satisfy in order for \( \varphi'(w) > 0 \) always?

(b) (5 marks): Explain the stochastic dynamics that \( n_t \) exhibits. Carefully explain how you could simulate the optimal labor supply choices.

(c) (3 marks): Suppose that the utility function is

\[
U(c, \ell) = \log(c) + \eta \log(\ell), \quad \eta > 0
\]

What pattern of labor supply would one observe given the fluctuations in \( w \)? How does your answer depend on the preference weight \( \eta \)? What stochastic dynamics does this imply for consumption? Explain your answers.

(d) (5 marks): Suppose instead that the utility function is

\[
U(c, \ell) = \log [c - v(1 - \ell)], \quad v(1 - \ell) = \frac{(1 - \ell)^{1+\gamma}}{1 + \gamma}, \quad \gamma > 0
\]
What pattern of labor supply would one observe given the fluctuations in $w$? How does your answer depend on the parameter $\gamma$? What does $\gamma$ measure? What stochastic dynamics does this imply for consumption? Explain your answers.

(e). (12 marks): Let the utility function be as in part (d). Suppose that the Markov chain has 4 states with

$$w = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

and transition matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ 0 & p_{22} & p_{23} & p_{24} \\ 0 & 0 & p_{33} & p_{34} \\ 0 & 0 & p_{43} & p_{44} \end{pmatrix}$$

(each $0 < p_{ij} < 1$ unless otherwise indicated). Finally, the initial distribution is

$$\pi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve for the stationary distribution of wages. Compute the implied stationary distribution of labor supply. Explain how the stationary distributions depend on the transition probabilities and on the parameter $\gamma$. Suppose we have the numbers

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$$

$$P = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0.50 & 0.25 & 0.25 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.10 & 0.90 \end{pmatrix}$$

$$\gamma = 2$$

Compute the average and standard deviation of the wage and the labor supply in the stationary distribution.