ARCH-M MODELS

Economic theory holds that investors should be rewarded for taking risks. The ARCH-M (ARCH in mean) model provides an explicit link between the risk (conditional volatility) and the best forecast of a time series.

No such relationship holds for the ARMA-ARCH models. For example, suppose that $y_t$ is $AR(1)$ with $ARCH(q)$ errors. In this case, we have $y_t = a y_{t-1} + \varepsilon_t$ where $\varepsilon_t$ is $ARCH(q)$. The $\varepsilon_t$ are distributed as $\varepsilon_t \mid \psi_{t-1} \sim N(0, h_t)$, where $h_t = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$ and $\psi_{t-1}$ is the information set at time $t-1$. Suppose we want to forecast $y_t$ at time $t-1$. Regardless of the value of the conditional volatility (risk) $h_t$, the best forecast of $y_t$ is just the conditional mean,

$$\mu_t = E[y_t \mid \psi_{t-1}] = ay_{t-1},$$

so the reward (expected future value) does not depend on the risk.

The ARCH-M model is

$$y_t = C + \theta \sqrt{h_t} + \varepsilon_t,$$

where $\varepsilon_t$ is $ARCH(q)$ with $\varepsilon_t \mid \psi_{t-1} \sim N(0, h_t)$. The best forecast of $y_t$ given $\psi_{t-1}$ is the conditional mean

$$\mu_t = E[y_t \mid \psi_{t-1}] = C + \theta \sqrt{h_t},$$

which is an explicit function of the risk $h_t$. Note that the model implies that $\mu_t$ is proportional to $\sqrt{h_t}$ rather than $h_t$. This is sensible, since doubling all observations should double (not quadruple) the forecast. If $\theta$ is positive, then $\mu_t$ increases with $h_t$: the reward increases with the risk. Note, however, that the reward is not guaranteed: High volatility (large $h_t$) implies that $y_t$ is expected to be large, but not that $y_t$ is guaranteed to be large. So reward is measured here by the expected future value $\mu_t$, not by the actual future value $y_t$. 
We have shown that the ARCH-M model can be written as

\[ y_t = \mu_t + \varepsilon_t, \]

where

\[ \mu_t = C + \theta \sqrt{h_t}. \]

Note that \( \mu_t \) is the best forecast of \( y_t \) which could have been made at time \( t-1 \), and \( \varepsilon_t \) is an ARCH(\( q \)) shock which was completely unforecastable from time \( t-1 \).

As long as \( \{\varepsilon_t\} \) is stationary, \( \{h_t\} \) must also be stationary, and hence \( \{\mu_t\} \) and \( \{y_t\} \) are stationary as well.

The \( \{y_t\} \) are not white noise, so they are linearly forecastable, but the best possible forecast \( \mu_t \) is a nonlinear one. On the other hand, the failure of the forecasts to depend directly on the autocorrelations may be a drawback of the model.

The parameters of the ARCH-M model can be estimated from data using the maximum likelihood method. The tsp command "arch(nar=q,mean) x c" will carry out the estimation. Here, q is the number of arch parameters, and x is the time series.

The ARCH-M model has been used to investigate the term structure of interest rates. Engle, Lilien and Robins (1977), in their paper which introduced the model, showed that there were significant ARCH-M effects (that is, \( \theta \) was statistically significant) for a series of excess returns on 6 month treasury bills compared to the return on two consecutive 3 month treasury bills. Here, \( \mu_t \) would represent the "risk premium" necessary to induce a risk-averse agent to hold the longer term asset.