Integrated Moving Averages

The Integrated Moving Average (IMA) is often a useful model for economic time series. It is related to "exponential smoothing", a simple method for forecasting time series, which will be discussed later in more detail. An integrated moving average is simply an ARIMA model with \( p = 0 \). That is, the IMA \((d,q)\) model is the same as the ARIMA \((0,d,q)\). The reason for the name "integrated moving average" should be clear: The IMA \((d,q)\) is a moving average which has been integrated \( d \) times. Here, we will study the simplest case, the IMA \((1,1)\), also known as ARIMA \((0,1,1)\). The model can be written as

\[
x_t - x_{t-1} = \varepsilon_t - a \varepsilon_{t-1},
\]

where \( a \) is between \(-1\) and \(1\) (because of the invertibility condition). Since \( d=1 \), the series \( \{x_t\} \) is nonstationary. So strictly speaking, the series has no mean. Nevertheless, it is useful to think of \( \{x_t\} \) as fluctuating about a local mean, \( \bar{x}_t \), which changes with \( t \). If we define \( \alpha = 1-a \), then it can be shown that \( \{x_t\} \) has the AR \((\infty)\) representation

\[
x_t = \alpha \sum_{k=1}^{\infty} (1-\alpha)^{k-1} x_{t-k} + \varepsilon_t, \tag{1}
\]

which is the same as saying that

\[
x_t = \bar{x}_{t-1} + \varepsilon_t, \tag{2}
\]

where

\[
\bar{x}_{t-1} = \alpha \sum_{k=1}^{\infty} (1-\alpha)^{k-1} x_{t-k}
\]

is the local mean at time \( t-1 \). We see from (3) that the local mean, \( \bar{x}_{t-1} \) is an Exponentially Weighted "Moving Average" (EWMA) of previous values of \( x_t \) with weights

\[
\alpha, \ \alpha(1-\alpha), \ \alpha(1-\alpha)^2, \ \alpha(1-\alpha)^3, \ldots,
\]

which decay towards zero geometrically, that is, exponentially fast. It is also interesting to note that these weights sum to 1, since

\[
\alpha [1+(1-\alpha)+(1-\alpha)^2+\cdots] = \alpha \left[ \frac{1}{1-(1-\alpha)} \right] = \frac{\alpha}{\alpha} = 1,
\]
where we have used the formula for the sum of a geometric series.

It can also be shown that \( \{x_t\} \) has the \( MA(\infty) \) representation

\[
x_t = \varepsilon_t + \alpha \sum_{k=1}^{\infty} \varepsilon_{t-k}.
\]  

(4)

Since from (2) we know that \( x_t = \bar{x}_{t-1} + \varepsilon_t \), it follows that

\[
\bar{x}_{t-1} = \alpha \sum_{k=1}^{\infty} \varepsilon_{t-k}.
\]

(5)

**Forecasting**

Since \( x_{t+1} = \bar{x}_t + \varepsilon_{t+1} \) from Equation (2), the best one-step forecast is \( \bar{x}_t \), the current local mean. This can be contrasted with the case of the random walk, where the best forecast is the most recent observation, \( x_t \). It can be shown that

\[
\bar{x}_t = \alpha x_t + (1 - \alpha) \bar{x}_{t-1}.
\]

(6)

Thus, each new local mean is a compromise (weighted average) of the previous local mean and the most recent observation. Formula (6) shows how the new observation \( x_t \) influences the value of the local mean, and is very useful for forming forecasts recursively in "real time": As new observations become available, we can simply update our local mean, and thereby obtain the new one-step forecast, without doing any long calculations.

To obtain \( h \)-step forecasts, we note from the \( MA(\infty) \) representation (4) that

\[
x_{t+h} = \varepsilon_{t+h} + \alpha \sum_{k=1}^{\infty} \varepsilon_{t+h-k}.
\]

so the best \( h \)-step forecast is

\[
f_{t,h} = \alpha (\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \cdots) = \alpha \sum_{k=1}^{\infty} \varepsilon_{t+1-k} = \bar{x}_t,
\]

where we have used equation (5) for the last step. Thus, we have shown that

\[
f_{t,h} = \bar{x}_t.
\]
so that the best forecast for any lead time is just the current local mean, $\bar{x}_t$.

To see this in another way, note from Equations (4) and (5) that

$$x_{t+h} - \bar{x}_t = e_{t+h} + \alpha \sum_{k=1}^{\infty} e_{t+h-k} - \alpha \sum_{k=1}^{\infty} e_{t+1-k}$$

$$= e_{t+h} + \alpha [e_{t+h-1} + e_{t+h-2} + \cdots + e_{t+1}]$$

$$= e_{t+h} + \alpha \sum_{k=1}^{h-1} e_{t+k}.$$ 

Thus,

$$x_{t+h} = \bar{x}_t + \alpha \sum_{k=1}^{h-1} e_{t+k} + e_{t+h},$$

so that as we move into the future from time $t$ (by letting $h$ increase), the process will diverge from the current local mean $\bar{x}_t$ according to the "random walk"

$$\alpha \sum_{k=1}^{h-1} e_{t+k} + e_{t+h},$$

which has zero mean and is not forecastable.