Recall that \( \{x_t\} \) is a martingale if \( E[x_{n+h} \mid x_n, x_{n-1}, \ldots] = x_n \) for all \( n \) and for all lead times \( h > 0 \). Actually, to establish that \( \{x_t\} \) is a martingale, one simply needs to prove the above formula for \( h = 1 \) since it can be shown that if it holds for \( h = 1 \) it must hold for all \( h > 0 \).

1) Suppose \( x_t = x_{t-1} + \varepsilon_t \) where \( \varepsilon_t = e_t + \beta e_{t-1} \), \( \beta \neq 0 \), and \( \{e_t\} \) is strict white noise.
   
a) What is the best linear predictor of \( x_{n+1} \) based on \( x_n, x_{n-1}, \ldots \) ? Justify your answer.

b) What is the best possible predictor of \( x_{n+1} \) based on \( x_n, x_{n-1}, \ldots \) ? Justify your answer.

c) Compare your answers to a) and b) to decide whether \( \{x_t\} \) is a martingale. (Keep in mind the discussion at the top of this handout).

2) Suppose \( x_t = \alpha x_{t-1} + e_t \) where \( \{e_t\} \) is strict white noise.
   
a) If \( |\alpha| < 1 \), is \( \{x_t\} \) a martingale? Justify your answer.

b) If \( \alpha = 1 \), is \( \{x_t\} \) a martingale? Justify your answer.

3) Suppose \( \{\varepsilon_t\} \) are martingale differences. Suppose we "integrate" \( \{\varepsilon_t\} \) to obtain a series \( \{y_t\} \).
   Specifically, define \( y_1 = \varepsilon_1, y_2 = \varepsilon_1 + \varepsilon_2, \) etcetera.
   
a) Show that \( y_t = y_{t-1} + \varepsilon_t \).

b) Use the results of a) to show that \( \{y_t\} \) is a martingale. (Thus, integrating a martingale difference yields a martingale.)