THE BISPECTRUM AND TESTS FOR NONLINEARITY

The Bispectrum

Whereas the spectrum provides a decomposition of the second moment into contributions from individual component frequencies, the bispectrum provides a similar decomposition of the third moment. First, consider the spectral density. This is the function $f(\lambda)$ such that

$$C(r) = \int_{-\pi}^{\pi} e^{i\lambda \lambda} f(\lambda) d\lambda.$$  \hspace{1cm} (1)

As long as the autocovariances $C(r) = E[x_t x_{t+r}]$ do not go to zero too slowly, we can invert this to give

$$f(\lambda) = \frac{1}{2\pi} \sum_r C(r) \exp(-i\lambda r),$$

so that the spectral density is the Fourier transform of the autocovariance function. Using the spectral representation, we find that

$$C(r) = E[x_t x_{t+r}] = \int \int e^{i\lambda_1 \lambda_2} d\tilde{Z}(\lambda_1) \int \int e^{i\lambda_1 \lambda_2} d\tilde{Z}(\lambda_2) = \int e^{i\lambda_1 \lambda_2} e^{i\lambda_1 \lambda_2} E[d\tilde{Z}(\lambda_1) d\tilde{Z}(\lambda_2)].$$

Since the left-hand side does not depend on $t$, neither does the right-hand side. Thus, we must have $E[d\tilde{Z}(\lambda_1) d\tilde{Z}(\lambda_2)] = 0$ unless $\lambda_1 + \lambda_2 = 0$. Therefore,

$$C(r) = \int_{\lambda = -\pi}^{\pi} e^{i\lambda r} E[d\tilde{Z}(-\lambda) d\tilde{Z}(\lambda)].$$

Comparing this with (1), we see that $f(\lambda) d\lambda = E[d\tilde{Z}(-\lambda) d\tilde{Z}(\lambda)]$. Thus, $f(\lambda) d\lambda$ is the joint second moment of the components of $x_t$ from two frequencies which sum to zero, one of which is $\lambda$. Since from (1) we have

$$C(0) = \int_{-\pi}^{\pi} f(\lambda) d\lambda,$$

we can also think of $f(\lambda) d\lambda$ as the contribution to the variance of $x_t$ from frequency $\lambda$.

The **bispectrum** $f(\lambda_1, \lambda_2)$ is the contribution to the *third* moment of $\{x_t\}$ from the *pair* of frequencies $\lambda_1, \lambda_2$. Consider the third order moment function
\[
C(r,s) = E[x_t x_{t+r} x_{t+s}] ,
\]
assumed not to depend on \( t \). The bispectrum is defined by

\[
C(r,s) = \frac{1}{(2\pi)^3} \sum_r \sum_s e^{i\lambda_1 r} e^{i\lambda_2 s} f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 .
\]  \hspace{1cm} (2)

This can be inverted (under suitable conditions) to give

\[
f(\lambda_1, \lambda_2) = \frac{1}{(2\pi)^3} \sum_r \sum_s C(r,s) e^{-i\lambda_1 r} e^{-i\lambda_2 s} ,
\]
so that the bispectrum is the (two dimensional) Fourier transform of \( C(\cdot, \cdot) \). Using the spectral representation, we have

\[
C(r,s) = \frac{1}{(2\pi)^3} \sum_r \sum_s e^{i\lambda_1 r} e^{i\lambda_2 s} E[\int dZ(\lambda_1) \int dZ(\lambda_2) e^{i\lambda_1 r} e^{i\lambda_2 s}] .
\]

Since the lefthand side does not depend on \( t \), neither does the righthand side, so

\[
E[dZ(\lambda_1) dZ(\lambda_2)] = 0
\]

unless \( \lambda + \lambda_1 + \lambda_2 = 0 \), and therefore

\[
C(r,s) = \frac{1}{(2\pi)^3} \sum_r \sum_s e^{i(\lambda_1 + \lambda_2) t} E[dZ((-\lambda_1 + \lambda_2)) dZ(\lambda_1) dZ(\lambda_2)] .
\]

Comparison with (2) reveals that

\[
f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 = E[dZ((-\lambda_1 + \lambda_2)) dZ(\lambda_1) dZ(\lambda_2)] ,
\]
so that \( f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \) is the joint third moment of the components of \( x_t \) from three frequencies which sum to zero, two of which are \( \lambda_1 \), \( \lambda_2 \). Since from (2) we have

\[
E[x_t^3] = \frac{1}{(2\pi)^3} \sum_r \sum_s f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 ,
\]
we can also think of \( f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \) as the contribution to \( E[x_t^3] \) from frequencies \( (\lambda_1, \lambda_2) \). If \( \{x_t\} \) is Gaussian, then \( C(r,s) = 0 \) for all \( r, s \), and the bispectrum will be identically equal to zero.
Subba Rao’s Test for Nonlinearity and Non-Gaussianity

Although many time series observed in practice may be nonlinear, it would be unrealistic to assume that the data were generated exactly by one of the known parametric models. These models (Bilinear, TAR, GARCH, etc.) are merely approximations which will sometimes be useful. The question of whether the series is nonlinear, however, seems more basic. By considering the bispectrum, and perhaps higher order spectra, it is possible to construct tests for nonlinearity without specifying any finite-parameter models, either under the null hypothesis or the alternative hypothesis. The null hypothesis (of linearity) is that \( x_t = \sum a_u e_{t-u} \), where the \( \{ e_t \} \) are iid. It can be shown (homework) that if \( \{ x_t \} \) is linear, then the quantity

\[
T(\lambda_1, \lambda_2) = \frac{|f(\lambda_1, \lambda_2)|^2}{f(\lambda_1) f(\lambda_2) f(\lambda_1 + \lambda_2)}
\]

will be constant, \( T(\lambda_1, \lambda_2) = \frac{\mu^2}{2\pi} \frac{1}{(\sigma^2)^3} \), where \( \mu_3 = E[e_t^3] \). Furthermore, if \( \{ x_t \} \) is Gaussian, then it is linear with \( \mu_3 = 0 \), so \( T(\lambda_1, \lambda_2) \) (and \( f(\lambda_1, \lambda_2) \)) will be identically equal to zero.

The first stage of Subba Rao’s test examines whether the series is Gaussian, by testing whether the estimated bispectrum over a grid of frequencies is significantly different from zero. If it is (as measured by a test statistic related to Hotelling’s \( T^2 \)), then the series is declared to be non Gaussian, and we can ask whether the process is linear or not. If \( \{ x_t \} \) is linear, then \( T(\lambda_1, \lambda_2) \) will be constant. So the second stage of Subba Rao’s test examines whether the sample version of \( T(\lambda_1, \lambda_2) \) is significantly non-constant, over a grid of frequencies. If it is, we declare the series to be nonlinear.

The advantage of Subba Rao’s test is that it is nonparametric. A drawback of the test is that, depending on exactly how we estimate the spectrum and bispectrum, our conclusions about Gaussianity and linearity of a given observed series may change. A more serious problem is that not all nonlinear processes will have a nonzero and nonconstant bispectrum. For example, the bispectrum of an ARCH process is identically zero. Thus Subba Rao’s test will have no chance of detecting nonlinearity (or even non Gaussianity) in this case. On the other hand, we are not likely to ever see a time series which is exactly ARCH. Furthermore, it is possible to generalize Subba Rao’s test to higher order spectra. For
example, the third-order spectrum ("trispectrum") of an ARCH process is nonzero (see Milhoj), while the third-order spectrum of a Gaussian process is identically zero.

Subba Rao’s test reveals that two of the data sets which traditionally have been analyzed in the time series literature using ARMA models (the sunspot series and the Canadian Lynx series) actually seem to be nonlinear.