8: FORMULAS INVOLVING THE PERIODOGRAM 
AND SAMPLE AUTOCOVARIANCES

Define the lag-$r$ sample autocovariance by

$$
\hat{c}_r = \frac{1}{n} \sum_{t=|r|}^{n-1} x_t x_{t-|r|}
$$

for $|r| < n$. Note that $\hat{c}_{-r} = \hat{c}_r$. The $\{\hat{c}_r\}$ sequence is very important in the time-domain analysis of time series (e.g., Box-Jenkins methods), and also plays an important role in frequency domain methods. Interestingly, it can be shown that the periodogram is the Fourier transform of the $\{\hat{c}_r\}$ sequence:

$$
I(\omega) = \frac{1}{2\pi} \sum_{|r| < n} \hat{c}_r \exp(-ir \omega)
$$

Proof:

$$
I(\omega) = \frac{1}{2\pi n} \left| \sum_{t=0}^{n-1} x_t \exp(-i\omega t) \right|^2 = \frac{1}{2\pi n} \sum_{t=0}^{n-1} \sum_{u=0}^{n-1} x_t x_u \exp\{-i(t-u)\omega\}
$$

Making the change of variables $v = t-u$, and rearranging terms (a diagram of the range of summation will be helpful), we obtain

$$
I(\omega) = \frac{1}{2\pi n} \sum_{u=0}^{n-1} \sum_{v=-u}^{n-1-u} x_{v+u} x_u \exp(-i\omega v)
$$

$$
= \frac{1}{2\pi n} \sum_{v=0}^{n-1} \sum_{u=0}^{n-1-v} x_{v+u} x_u \exp(-i\omega v) + \frac{1}{2\pi n} \sum_{v=-(n-1)}^{-1} \sum_{u=-v}^{n-1} x_{v+u} x_u \exp(-i\omega v)
$$

$$
= \frac{1}{2\pi} \left[ \sum_{v=0}^{n-1} \hat{c}_v \exp(-i\omega v) \right] + \frac{1}{2\pi} \sum_{v=-(n-1)}^{-1} \sum_{u=0,v+u}^{n-1} x_{u-v} x_u \exp(-i\omega v)
$$

$$
= \frac{1}{2\pi} \sum_{|r| < n} \hat{c}_r \exp(-ir \omega)
$$

Another interesting formula is

$$
\hat{c}_r = \int_{-\pi}^{\pi} I(\omega) \exp(ir \omega)
$$

showing that $\hat{c}_r$ is the integral Fourier transform of the periodogram.
Proof:

\[
\hat{c}_r = \frac{2\pi}{n} \sum_{j=0}^{n'-1} I(\omega'_j) \exp(i r \omega'_j), \quad \omega'_j = \frac{2\pi j}{n'}, \quad n' = 2n.
\]

Since \( I(\omega) = \frac{1}{2\pi} \sum \hat{c}_s \exp(-ir \omega) \), the \( \{ \hat{c}_r \} \) sequence determines the periodogram. Furthermore, the periodogram, if evaluated on a sufficiently fine grid, completely determines the covariance sequence.

We have

\[
\hat{c}_r = \frac{2\pi}{n} \sum_{j=0}^{n'-1} I(\omega'_j) \exp(i r \omega'_j), \quad \omega'_j = \frac{2\pi j}{n'}, \quad n' = 2n.
\]

Proof: Append \( n \) zeros to \( x_0, \ldots, x_{n-1} \) to obtain \( y_0, \ldots, y_{n'-1} \). We have \( \hat{c}_{y,r} = \frac{1}{2} \hat{c}_{x,r} \) (\( |r| < n \)), \( \hat{c}_{y,r} = 0 \) (\( |r| \geq n \)), and \( I_r(\omega) = \frac{1}{2} I_x(\omega) \). Given \( r \) with \( |r| < n \), the RHS of the formula to be proved is

\[
\frac{2\pi}{n} \sum_{j=0}^{n'-1} I_r(\omega'_j) \exp(ir \omega'_j) = \frac{2\pi}{2n} \sum_{j=0}^{n'-1} 2I_r(\omega'_j) \exp(ir \omega'_j) = \frac{2\pi}{n} \sum_{j=0}^{n'-1} \left[ \frac{1}{2\pi} \sum_{|s| < n'} \hat{c}_{y,s} \exp(-is \omega'_j) \right] \exp(ir \omega'_j)
\]

\[
= \sum_{|s| < n'} \hat{c}_{y,s} \frac{1}{n} \sum_{j=0}^{n'-1} \exp(i (r-s) \omega'_j) = \sum_{|s| < n} \hat{c}_{x,s} \frac{1}{n} \sum_{j=0}^{n'-1} \exp(i j \omega'_{r-s})
\]

\[
= \sum_{|s| < n} \hat{c}_{x,s} \cdot \text{Indicator}(r-s = 0 \mod n') = \hat{c}_{x,r} = \hat{c}_r.
\]

As a consequence of the above identity, the \( \{ \hat{c}_r \} \) sequence may be computed in \( O(n \log n) \) using the FFT.