9: THE SPECTRUM

The spectrum is a key quantity in the frequency domain analysis of stationary time series. The true meaning of the spectrum will become clear only later, but here we will give some preliminary definitions. We now assume that $x_0, \ldots, x_{n-1}$ are random variables, so that our data form a piece of a realization of a stochastic process. This assumption allows us to formalize the statistical analysis of time series. It has not been needed in most of the previous sections. We also assume that the process $\{X_t\}_{t=-\infty}^{\infty}$ is weakly stationary: (i) $E[X_t]$ is a finite constant, and (ii) $E[X_t X_u]$ depends only on $t-u$.

For simplicity, we will take the constant mean value to be zero.

A key quantity in the time domain analysis of stationary time series is the theoretical autocovariance sequence $\{c_r\}$, where

$$c_r = \text{Cov}(X_t, X_{t+r}) = \text{E}[X_t X_{t+r}] = \text{E}[X_t X_{t-r}] = c_{-r} .$$

The traditional (although biased) estimate of $c_r$ is the sample autocovariance

$$\hat{c}_r = \frac{1}{n-r} \sum_{t=r}^{n-1} x_t x_{t-r} .$$

The $\{c_r\}$ sequence is nonnegative definite, in the sense that for any constants $a_1, \ldots, a_k$,

$$\sum_{r,s} a_r c_{r-s} a_s \geq 0.$$ 

**Proof:**

$$0 \leq \text{Var} \sum_r a_r X_{t-r} = \text{E} [\sum_r a_r X_{t-r}]^2 = \text{E} [\sum_r a_r X_{t-r} \sum_s a_s X_{t-s}] = \sum_{r,s} a_r c_{r-s} a_s .$$

Since $\{c_r\}$ is nonnegative definite, Herglotz’ Theorem (from analysis) guarantees the existence of a non-decreasing spectral distribution function $F(\omega)$ defined on $[-\pi, \pi]$, such that $\{c_r\}$ is the Fourier transform of the measure corresponding to $F$:

$$c_r = \int_{-\pi}^{\pi} \exp(-ir \omega) dF(\omega) .$$

If $F(\omega)$ has derivative $f(\omega) = F'(\omega)$, then $f(\omega)$ is called the spectral density (or sometimes "the spectrum"), and we have

$$c_r = \int_{-\pi}^{\pi} \exp(-ir \omega) f(\omega) d\omega .$$
Assuming weak conditions on \( \{ c_r \} \) (for example, \( \sum_r |c_r| < \infty \)), the above Fourier transform can be inverted to give

\[
f(\omega) = \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} c_r \exp(ir\omega) .
\]

Thus, the covariance sequence is the (integral) Fourier transform of the spectrum, and the spectrum is the (discrete) Fourier transform of the covariance sequence. Note the similarity between the above two equations and their finite-sample analogs

\[
\hat{c}_r = \frac{\pi}{\int_{-\pi}^{\pi} \exp(-ir\omega)I(\omega)d\omega} , \quad I(\omega) = \frac{1}{2\pi} \sum_{|r|<\pi} \hat{c}_r \exp(ir\omega) .
\]

The second of these suggests that \( I(\omega) \) might be a useful estimate of \( f(\omega) \). In fact, however, the unmodified periodogram is an inconsistent (and hence a very poor) estimate. To see this, consider the simplest of all processes, Gaussian white noise: \( x_0, \ldots, x_{n-1} \) are iid \( N(0, \sigma^2) \). The spectrum is then constant, \( f(\omega) \equiv \frac{1}{2\pi} c_0 = \frac{\sigma^2}{2\pi} \). As was shown earlier, if \( \omega \) is a Fourier frequency,

\[
\sum \chi_i \cos \omega t , \quad \sum \chi_i \sin \omega t
\]

are uncorrelated, mean 0, variance \( \frac{n}{2} \sigma^2 \). Thus,

\[
I(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{2} \chi_i^2 = \text{Exponential (Mean} = \frac{\sigma^2}{2\pi}) = f(\omega) \frac{1}{2} \chi_i^2 .
\]

We see that \( I(\omega) \) does not converge in probability to \( f(\omega) \). Instead, \( I(\omega) \) converges in distribution to \( f(\omega) \frac{1}{2} \chi_i^2 \). \( I(\omega) \) is not a consistent estimator of \( f(\omega) \). Also, \( I(\omega_j) \) and \( I(\omega_k) \) are independent, so the periodogram as a function of \( \omega \) is erratic (rough), even if the underlying spectrum is smooth.

As will be shown later, the relation

\[
I(\omega_j) = f(\omega_j) \frac{1}{2} \chi_i^2
\]

is approximately true for a very wide class of series, not just white noise. Thus, \( E[I(\omega)] = f(\omega) \): The periodogram is asymptotically unbiased. This justifies Bloomfield’s heuristic definition of the spectrum.
as the aspects of the periodogram which exhibit statistical regularity.

A reasonable way to estimate \( f(\omega) \) is to smooth \( I(\omega) \), for example

\[
\hat{f}(\omega_j) = \sum_k g_k I_{j-k}.
\]

This and other methods will be discussed later.