HOMEWORK 10

1) If \( \{ \varepsilon_t \} \) is GARCH \((p,q)\), use equations (1) and (2) in the notes to show that

a) \( E[\varepsilon_t]=0 \)

b) \( E[\varepsilon_s \varepsilon_t]=0 \) if \( s<t \)

c) \( \text{var}[\varepsilon_t]=E[h_t] \)

d) If \( \{ \varepsilon_t \} \) is weakly stationary, then \( \text{var}[\varepsilon_t]=\frac{\omega}{1-\sum_{i=1}^{q}\alpha_i-\sum_{j=1}^{p}\beta_j} \). It follows that if \( \{ \varepsilon_t \} \) is weakly stationary then \( \sum_{i=1}^{q}\alpha_i+\sum_{j=1}^{p}\beta_j<1 \).

2) If \( \{ x_t \} \) is ARMA with GARCH errors, show that

a) The optimal one-step predictor \( x_{t+1} \) is a linear combination of \( x_t, x_{t-1}, \ldots \).

b) \( \text{var}[x_t|x_{t-1}]=h_t \). (So the conditional variances of \( x_t \) and \( \varepsilon_t \) are the same.)

c) \( E[x_t]=0 \)

d) \( \text{corr}(x_t,x_{t+k}) \) does not depend on \( \alpha_1,\ldots,\alpha_q,\beta_1,\ldots,\beta_p \). (Hint: Use the spectral representation to get the MA(\( \infty \)) representation for \( \{ x_t \} \).) It follows that, except for a multiplicative constant, the covariance sequence and spectral density of \( \{ x_t \} \) are the same as in the linear ARMA case.

3) For an ARCH\((q)\) model, adding \( \eta_t=\varepsilon_t^2-h_t \) to both sides of Equation (2) in the handout yields

\[
\varepsilon_t^2=\omega+\sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2+\eta_t .
\]

Assuming that \( \{ \varepsilon_t^2 \} \) has finite variance, use the equation above to prove that \( \{ \varepsilon_t^2 \} \) obeys an AR\((q)\) model with parameters \( \alpha_1,\ldots,\alpha_q \). You need to prove that \( \{ \eta_t \} \) is zero mean white noise, and that
\[ \text{Cov}[\varepsilon_t^2, \eta_t] = 0 \text{ if } s < t. \]

4) For the General Motors (GM) data,
   a) Identify an ARCH model using graphical methods.
   b) Estimate the parameters by the Yule-Walker method.
   c) Based on the estimated model, what was the conditional volatility on the day before the crash (Friday, October 16)? Use this to construct a 99% forecast interval for the October 19 return. Did the actual October 19 return fall within the forecast interval?
   d) What was the conditional volatility on the day of the crash (Monday, October 19)? Compare this to the conditional volatility from part c).

5) Simulate two ARCH (1) models with \((100, 0.25, 0.35)\) and \((100, 0.25, 0.85)\), respectively. Plot both time series, and describe what you see. Look at the sample autocorrelation function and partial autocorrelation function of both the original and squared series, and describe what you see. Compute the Yule-Walker and maximum likelihood estimates for both series. For each series, how good is the agreement between the two types of estimators? (Note: For the MLE, you will need to optimize a two-dimensional function. Restrict yourself to the \(100 \times 100\) grid \((\omega, \alpha_1) = (i - 0.5, j - 0.5)/100\) with \(i, j = 1, \ldots, 100\).

6) Consider the ARCH (1) process \(\{\varepsilon_t\}\), defined by \(\varepsilon_t = \varepsilon_t \sqrt{h_t}\), where the \(e_t\) are iid standard normal.
   a) Use the Yule-Walker equations to show that \(\gamma_t = \alpha_1 \gamma_{t0}\), where \(\gamma_t = \text{cov}(\varepsilon_t^2, \varepsilon_{t-1}^2)\), and \(\alpha_1\) is the ARCH parameter.
   b) Show that \(E[\varepsilon_t^4] = \left[6 \alpha_2^2 + 6 \text{var}(\varepsilon_t) \alpha_1^2 \right]/(1 - 3\alpha_1^2)\). Hint: Remember that \(\epsilon_t\) is independent of \(h_t\). You will also need to use the fact that \(E[\varepsilon_t^4] = 3\).
   c) Use parts a and b to verify that
\[ \frac{\gamma_1(1-\alpha_1)^2}{\omega^2} \alpha_2 = \frac{2\alpha_1'}{1-3\alpha_1''}, \]

as stated in the notes. Hint: You will need an expression for \( \gamma_0 \).

7) Use the formula \( E[y_t] = (I-A)^{-1}b \) given in the notes to prove (again) that if \( \{\varepsilon_t\} \) is \( ARCH(q) \) then

\[ \text{var} [\varepsilon_t] = \omega'(1 - \sum_{i=1}^{q} \alpha_i). \]

8) Show that the set of eigenvalues of \( A \) consists of 0, together with all roots of \( Q(z) \).

9) Use the formula for the roots of a quadratic equation to prove directly that for the \( ARCH(2) \), Engle’s and Milhoj’s conditions for stationarity are equivalent. In other words, show that \( \alpha_1 + \alpha_2 < 1 \) if and only if all roots of \( P(z) = 1 - \alpha_1 z - \alpha_2 z^2 \) are outside the unit circle.

10) In the notes, we proved that if all roots of \( P(z) \) are outside the unit circle, then the \( ARCH(q) \) process \( \{\varepsilon_t\} \) is weakly stationary. Use the theorem on Milhoj’s condition, and the results of earlier homework problems, to show that if \( \{\varepsilon_t\} \) is weakly stationary, then all roots of \( P(z) \) are outside the unit circle. Thus, Engle’s condition, Milhoj’s condition and weak stationarity are all equivalent.