The Projection Theorem for Hilbert Spaces

Suppose $H$ is a Hilbert space. Thus, $H$ is a vector space with an inner product $(\cdot, \cdot)$ satisfying the properties given in Handout 10, page 2. We say that vectors $x, y$ in $H$ are orthogonal if $(x, y) = 0$. The norm of a vector $x$ in $H$ is defined by $\|x\| = (x, x)^{1/2}$.

Let $M$ be a subspace of $H$. ($M$ is a vector space, and any element of $M$ is an element of $H$).

Let $x$ be any vector in $H$ but not in $M$. We want to find the vector $z$ in $M$ which is closest to $x$ in the sense that

$$\|x - z\| = \min_{y \in M} \|x - y\|.$$

This means that $\|x - z\| \leq \|x - y\|$ for any $y$ in $M$. In other words, we want to find the vector $z$ in the subspace $M$ which is at least as close to the given vector $x$ as any other $y$ in $M$. Such a vector $z$ is called the orthogonal projection of $x$ on $M$.

We are going to show that if $z$ satisfies the following two conditions, then $\|x - z\| = \min_{y \in M} \|x - y\|$. so that $z$ is the orthogonal projection of $x$ on $M$. The conditions are

a) $z \in M$

b) $(x - z) \perp M$.

Condition b) means that $x - z$ is orthogonal to any element of $M$.

In Problems 1-4, suppose that $z$ does satisfy conditions a) and b) above, and suppose that $y$ is any element of $M$.

1) Show that $z - y$ is in $M$.

2) Show that $(x - z) \perp M$.

3) Show that $\|x - y\|^2 = \|x - z\|^2 + \|z - y\|^2$. 
4) Show that \( \|x - z\| = \min_{y \in M} \|x - y\| \), so that \( z \) is indeed the orthogonal projection of \( x \) on \( M \).

5) Consider the special case where \( H = M^X \), the Hilbert space generated by linear combinations of the weakly stationary zero mean process \( \{X_t\} \) with inner product \( (X, Y) = E[XY] \). Take \( M = M^X_t \), the linear subspace generated by the elements \( X_s \) for \( s \leq t \). Take the vector \( x \) to be \( X_{t+\nu} \). Then explain why the problem of finding the orthogonal projection of \( x \) on \( M \) reduces to the linear prediction problem. What does the orthogonal projection of \( X_{t+\nu} \) on \( M^X_t \) minimize?