19. CONFIDENCE INTERVALS FOR THE MEAN; KNOWN VARIANCE

We assume here that the population variance $\sigma^2$ is known. This is an unrealistic assumption, but it allows us to give a simplified presentation which reveals many of the important issues, and prepares us to solve the real problem, where $\sigma^2$ is unknown. (Next handout).

We want a confidence interval for the population mean, $\mu$, based on $n$ observations.

There is some small probability ($\alpha$) that the method of constructing confidence intervals will fail. We can control this probability, that is, we can select any value for $\alpha$ we want. Traditionally, $\alpha$ is taken to be either 0.05 or 0.01. We refer to $1 - \alpha$ as the confidence level. This is the proportion of the time that the confidence interval contains the parameter, in repeated sampling.

Sincich uses the name “confidence coefficient” for $1 - \alpha$.

• Let $z_{\alpha/2}$ denote the $z$ value such that the area to its right under the standard normal curve is $\alpha/2$.

Eg: For $\alpha = 0.05$ we get $z_{0.025} = 1.96$.

So, 95% of the time a normal will be in the range $(\mu - 1.96\sigma, \mu + 1.96\sigma)$.

In the Empirical Rule, we approximated the 1.96 by 2.

For $\alpha = 0.01$ we get $z_{0.005} = 2.576$.

• The interval $\bar{X} \pm z_{\alpha/2}\sigma_\bar{X}$ is a CI for $\mu$ with confidence level $1 - \alpha$.

Note that the interval extends from $\bar{X} - z_{\alpha/2}\sigma_\bar{X}$ to $\bar{X} + z_{\alpha/2}\sigma_\bar{X}$.
Here’s why the formula above works. First, we suppose that the Central Limit Theorem allows us to assume that $\bar{X}$ is normal.

Let’s convert $\bar{X}$ into its own $z$-score,

$$Z = \frac{\bar{X} - \mu}{\sigma_x}$$

We have standardized $\bar{X}$ by subtracting its own mean, $\mu_x = \mu$, and dividing by its own standard error, $\sigma_x = \frac{\sigma}{\sqrt{n}}$.

Since $\bar{X}$ is normal, $Z$ is standard normal.

$Z$ measures how many standard errors $\bar{X}$ falls from $\mu$.

The probability that $\bar{X}$ falls more than $z_{\alpha/2}$ standard errors from $\mu$ is

$$\text{Prob}\{Z < -z_{\alpha/2}\} + \text{Prob}\{Z > z_{\alpha/2}\} = \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$$

Therefore,

$$1 - \alpha = \text{Prob}\{\bar{X} \text{ falls within } z_{\alpha/2} \text{ standard errors of } \mu\}$$

$$= \text{Prob}\{\mu \text{ is within } z_{\alpha/2} \sigma_x \text{ of } \bar{X}\}$$

$$= \text{Prob}\{\mu \text{ is between } \bar{X} - z_{\alpha/2} \sigma_x \text{ and } \bar{X} + z_{\alpha/2} \sigma_x\}.$$

So $\bar{X} \pm z_{\alpha/2} \sigma_x$ is a CI for $\mu$ with confidence level $1 - \alpha$.

**Interpretation of Confidence Intervals**

As stated earlier, the confidence interval $\bar{X} \pm z_{\alpha/2} \sigma_x$ will cover $\mu$ with probability $1 - \alpha$. There is a subtle difficulty in the practical interpretation of the results, however, as demonstrated in the following example.

**Eg 1**: In the Pepsi example, we had $n = 100$, $\sigma = 0.05$ and $\bar{x} = 1.985$. The 95% confidence interval estimator for $\mu$ is

$$\bar{X} \pm 1.96 \frac{\sigma}{\sqrt{n}}$$

that is, $\bar{X} \pm 0.01$

Plugging in $\bar{x} = 1.985$, we obtain the 95% confidence interval estimate $(1.975, 1.995)$, rounded to three decimal places.
Which of the following statements is true?

a) There is a 95% chance that \( \mu \) is between 1.975 and 1.995.

b) \( \mu \) will be between 1.975 and 1.995 95% of the time.

c) In 95% of all future samples, \( \bar{x} \) will be between 1.975 and 1.995.

d) \( \mu \) is between 1.975 and 1.995.

e) None of the above.

• Warning: The practical interpretation of confidence intervals is extremely tricky. The difficulty has to do with the distinction between an estimator and an estimate.

An estimator is a random variable whose value depends on a sample not yet taken (eg: \( X \)).

An estimate is the value actually taken by the estimator for a given sample (eg: \( \bar{x} = 1.985 \)).

• The CI is an interval estimator. It has random endpoints. After the sample is taken, its endpoints take on specific values, yielding an interval estimate.

• Since \( \mu \) is an (unknown) constant, and since the endpoints of the CI estimate are fixed numbers (eg: 1.975, 1.995), it makes no sense to talk about the probability that the CI estimate contains \( \mu \). Either it does or it doesn’t, and we may never find out which of these events has occurred.

Instead, it is the CI estimator which contains \( \mu \) with probability \( 1 - \alpha \).

The estimator has random endpoints,

\[ \bar{X} - z_{\alpha/2} \sigma, \bar{X} + z_{\alpha/2} \sigma \] .

The word “probability” refers to the long-run proportion of the time that these random endpoints will contain the true mean \( \mu \), assuming a large number of repetitions of the experiment of collecting a random sample and constructing the CI.

Thus the confidence level \( 1 - \alpha \) refers to the process of constructing confidence intervals, not to the particular CI estimate obtained from the given sample.

A correct (but not very satisfying) answer to the multiple choice problem is

f) We can’t really say anything about the particular interval estimate we got for this sample (1.975, 1.995), but the confidence interval estimator \( \bar{X} \pm 0.01 \) will cover \( \mu \) in 95% of all random samples which can be collected.
• Unfortunately, in practice we have only one sample. So what good is a probability statement referring to all samples which might have been taken?

1) You can think of $1 - \alpha$ as an overall success rate. If you compute many 95% confidence intervals over your lifetime, and if the required assumptions are satisfied for each one, then approximately 95% of these confidence intervals will contain their respective population means. Unfortunately, you may never know which ones were wrong.

2) Even though we can’t talk about the probability that the given CI estimate contains $\mu$, if we had to bet on it, we would say that the odds are 19 to 1 that our given 95% CI estimate fails to cover $\mu$. In the long run, in a lifetime of gambling on confidence intervals, you would win 19 out of 20 bets that the CI covered $\mu$.

See Web demo on confidence intervals based on repeated samples at:

http://www.stat.sc.edu/~west/javahtml/ConfidenceInterval.html

Try changing the value of alpha and watch the CIs widen or narrow.

Remember: In real life, we have only one sample, so only one CI. It might be one of the 5% of all CIs which will fail to contain the mean. We don’t know. But we have 95% confidence in the statistical procedure, since only 1 out of 20 such intervals will fail in the long run.

Why Don’t We Use 100% Confidence Intervals?

Wouldn’t it be better to be right 100% of the time rather than 95% of the time? Not necessarily, when it comes to confidence intervals. The problem is that (for given $n$ and $\sigma$), the smaller we make $\alpha$ the wider the CI becomes.

Eg: “The analysis division is 100% sure that the typical consumer will be willing to spend between $0$ and $35$ million per tube of toothpaste.”

A 95% CI is usually much more informative:

Eg: “The analysis division is 95% sure that the typical consumer will be willing to spend between $2.26$ and $2.74$ for a tube of toothpaste.”