Quiz #3
December 2014

Please write your name below. Then complete the exam in the space provided. There are THREE questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

(Name and signature)

1. Short answers. Provide short answers to the following:
   (a) Explain how to compute the continuously-compounded yield $y^n_t$ from bond price(s). (10 points)
   (b) Explain how to compute the continuously-compounded forward rate $f^n_t$ from bond price(s). (10 points)
   (c) If $\{x_t\}$ is a stochastic process, what property would make it a martingale? (10 points)

   Solution:
   (a) Use $y^n_t = -n^{-1} \log q^n_t$.
   (b) Use $f^n_t = \log(q^n_t/q^{n+1}_t)$.
   (c) We would say $x$ is a martingale if $E_t(x_{t+1}) = x_t$.

2. Nonlinear dynamics. Consider the stochastic process
   \[ x_{t+1} = x_t w_{t+1} + \theta w_t, \]
   where $\{w_t\}$ is a sequence of independent normal random variables with means equal to zero and variances equal to one. Assume, as usual, that at date $t$ we know the current and past values of $x$ and $w$, but not the future values.
   (a) What is the distribution of $x_{t+1}$ at date $t$ — the one-period conditional distribution, in other words? (10 points)
   (b) Show that $x$ is Markov for some definition $z$ of the state. (10 points)
   (c) What is $E_t(x_{t+2})$? [Hint: Use the law of iterated expectations.] (10 points)
   (d) What is $\text{Var}_t(x_{t+2})$? [Comment: This is difficult, skip if you’re short of time.] (10 points)
3. Valuing dividend strips. A dividend strip is a claim to a single dividend $n$ periods in the future. We denote the price at date $t$ of the dividend paid at $t+n$ by $s^n_t$. The term structure of strip prices — the sequence $s^1_t, s^2_t, \ldots$ — can be approached with methods similar to those we used with bonds.

We’ll use the model

$$\log m_{t+1} = -\lambda^2/2 - z_t + \lambda w_{t+1}$$

$$z_t = (1-\varphi)\delta + \varphi z_{t-1} + \sigma w_t$$

$$\log d_t = \eta z_t.$$

Here $\log m_{t+1}$ is the pricing kernel, $z_t$ is a state variable, $w_t$ is one of a sequence of independent standard normal random variables, and $d_t$ is the dividend. The parameter $\eta$ controls the sensitivity of the dividend $d_t$ to the state $z_t$.

(a) What is the short rate $f^0_t = y^1_t$ in this model? (10 points)

(b) What is the price $s^1_t$ of next period’s dividend $d_{t+1}$? (10 points)

(c) Value prices of future dividends recursively. If prices are loglinear functions of the state,

$$\log s^n_t = C_n + D_n z_t,$$

how would you compute the coefficients $(C_n, D_n)$? (20 points)

(d) Derive the excess log return on the strip of maturity one,

$$\log d_{t+1} - \log s^1_t - f^0_t.$$

How does it vary with $\eta$? (10 points)
Solution:
(a) The short rate is
\[ f_0^t = -\log q_t^1 = -\log E_t(m_{t+1}) = z_t. \]
(b) The price the one-period strip is
\[ \log s_1^t = \log E_t(m_{t+1}d_{t+1}) = \log E_t[\exp(\log m_{t+1} + \log d_{t+1})]. \]
Thus we need
\[ \log m_{t+1} + \log d_{t+1} = -\lambda^2/2 + \eta(1 - \varphi)\delta + (\eta \varphi - 1)z_t + (\eta \sigma + \lambda)w_{t+1}. \]
The usual “mean plus variance over two” gives us
\[ \log s_1^t = (\eta \sigma + \lambda)^2/2 - \lambda^2/2 + \eta(1 - \varphi)\delta + (\eta \varphi - 1)z_t. \]
Thus we have \( C_1 = (\eta \sigma + \lambda)^2/2 - \lambda^2/2 + \eta(1 - \varphi)\delta \) and \( D_1 = (\eta \varphi - 1) \).
(c) Strip prices of higher maturity follow from \( s_{t+1}^n = E_t(m_{t+1} s_{t+1}^n). \) Given their loglinear form, we solve
\[ \log m_{t+1} + \log s_{t+1}^n = -\lambda^2/2 + C_n + D_n(1 - \varphi)\delta + (D_n \varphi - 1)z_t + (D_n \sigma + \lambda)\eta_{t+1}. \]
Then we have
\[ \log s_{t+1}^n = \log E_t(m_{t+1} s_{t+1}^n) \]
\[ = (D_n \sigma + \lambda)^2/2 - \lambda^2/2 + C_n + D_n(1 - \varphi)\delta + (D_n \varphi - 1)z_t \]
\[ = C_{n+1} + D_{n+1}z_t. \]
Lining up similar terms gives us recursions in the coefficients:
\[ C_{n+1} = (D_n \sigma + \lambda)^2/2 - \lambda^2/2 + C_n + D_n(1 - \varphi)\delta \]
\[ D_{n+1} = (D_n \varphi - 1). \]
We can start with \( (C_1, D_1) \) above, or note that a zero maturity strip gives us \( \log s_0^t = \log d_t = \eta z_t \), which gives us \( C_0 = 0 \) and \( D_0 = \eta \).
(d) The log excess return is
\[ \log d_{t+1} - \log s_1^t - f_1^0 = \eta[(1 - \varphi)\delta + \varphi z_t + \sigma w_{t+1} - (C_1 + D_1z_t) - z_t \]
\[ = \lambda^2/2 - (\eta \sigma + \lambda)^2/2 + \eta \sigma w_{t+1}. \]
The (log) risk premium is the (conditional) mean, which we can simplify:
\[ E_t(\log d_{t+1} - \log s_1^t - f_1^0) = \lambda^2/2 - (\eta \sigma + \lambda)^2/2 = -(\eta \sigma)^2/2 - \lambda \eta \sigma. \]
Thus the risk premium depends on three parameters: \( \lambda \), the sensitivity of the pricing kernel to risk in \( z \); \( \sigma \), the magnitude of this risk; and \( \eta \), the sensitivity of the dividend to the same risk.