Binomial Models 2

1. Interest rate caps
2. Black-Scholes and binomial valuation of caps
3. Swaps
4. Swaptions
5. Black-Scholes and binomial valuation of swaptions
6. Summary and final thoughts
1. Interest Rate Caps

- Terminology:
  - An interest rate cap pays the difference between a reference rate and the cap rate, if positive (a series of call options on an interest rate)
  - An interest rate floor pays the difference between the floor rate and a reference rate, if positive (a series of put options on an interest rate)
  - A caplet (floorlet) is a single payment in a cap (floor)
  - An interest rate collar is a long position in a cap plus a short position in a floor (it puts upper and lower bounds on floating interest payments)

- Example: a 5-year semiannual cap would typically pay the difference between 6-month LIBOR and the cap rate every six months starting in 12 months (the first payment is generally dropped) and ending in 60 months

- It’s convenient to measure time in periods between payments: \( t \) is trade date (“now”), payments occur at \( t + 2, t + 3, \) etc

- Day counts (which we ignore) follow LIBOR conventions

- Timing: if there are \( m \) payments per year and the notional principal is 100, a caplet’s cash flows of

\[
m^{-1}(Y_{t+j} - K)^+
\]

are observed at \( t + j \) but paid one period (six months?) later (think about how LIBOR is paid)
2. Black-Scholes Valuation of Caps

- The Black-Scholes formula for a caplet whose underlying rate is observed in \( j \) periods and paid in \( j + 1 \) periods is

\[
\text{Caplet Price} = m^{-1} \left[ b^{j+1} F^j \Phi(d_j) - b^{j+1} K \Phi(d_j - (jh)^{1/2}v) \right]
\]

\[
d_j = \frac{\log(F^j/K) + (jh)v^2/2}{(jh)^{1/2}v}
\]

- \( F^j = \) \( j \)-period forward rate
- \( K = \) cap rate
- \( j = \) number of periods until rate is observed
- \( j + 1 = \) number of periods until rate is paid
- \( m = \) number of payments per year
- \( h = 1/m = \) time between payments in years
- \( jh = \) number of years until rate is observed
- \( b^{j+1} = (j + 1)\)-period discount factor
- \( \Phi = \) cumulative normal distribution function
- \( v = \) annualized volatility

Comments:

- the only new issue here is the difference between when the rate is observed and when it’s paid
- the forward rate \( F \) follows the same day count and compounding convention as \( Y \)
- the value of a cap is the sum of the values of its component caplets
- presumption: the underlying rate is lognormal
2. Black-Scholes Valuation of Caps (continued)

- Numerical example

  - Consider the following interest rate data:

    | Period (j) | Disc Factor | Spot Rate | Fwd Rate | $F^{j-1}$ |
    |------------|-------------|-----------|----------|-----------|
    | 1          | 0.975365    | 4.989     | 5.052    |           |
    | 2          | 0.949999    | 5.129     | 5.340    |           |
    | 3          | 0.924837    | 5.209     | 5.442    |           |
    | 4          | 0.899541    | 5.294     | 5.624    |           |
    | 5          | 0.874550    | 5.362     | 5.715    |           |
    | 6          | 0.849939    | 5.420     | 5.791    |           |

    (as usual, the data are based on the quote sheet)
    Keep these numbers in mind — we’ll come back to them

  - Caplet prices for $K = 5.50\%$ are

    | Period (j + 1) | Volatility | Caplet Price | Cap Price |
    |----------------|------------|--------------|-----------|
    | 1              | none       | none         | none      |
    | 2              | 12.50      | 0.0578       | 0.0578    |
    | 3              | 15.00      | 0.1381       |           |
    | 4              | 16.50      | 0.2304       | 0.4264    |
    | 5              | 17.00      | 0.2847       |           |
    | 6              | 17.50      | 0.3305       | 1.0414    |

  - Comments:

    * caps are sums of caplets
    * you might want to work through some of these calculations, but don’t get bogged down
2. Black-Scholes Valuation of Caps (continued)

- Cap valuation:
  \[ \text{Cap Price} = m^{-1} \sum_j \left[ b^{j+1} F^j \Phi(d_j) - b^{j+1} K \Phi(d_j - (jh)^{1/2}v) \right] \]
  \[ d_j = \frac{\log(F^j/K) + (jh)v^2/2}{(jh)^{1/2}v} \]

  - These things vary with \( j \): forward rate \( F^j \), discount factor \( b^{j+1} \), “\( d_j \)”
  - These do not: volatility \( v \), cap rate \( K \)

- Implied volatility: find the value of \( v \) that generates the observed price
  - Our example: if 2-year 5.5% cap price is 0.4264, implied volatility is 15.03%
  - Comment: this is a composite of the volatilities of the caplets, which are not generally the same for all maturities (in our example, they are 12.50, 15.00, and 16.5)
3. Black-Derman-Toy Valuation of Caps

- The short rate tree for BDT model

```
<table>
<thead>
<tr>
<th>Rate</th>
<th>4.989</th>
<th>5.735</th>
<th>6.567</th>
<th>7.724</th>
<th>8.881</th>
<th>10.254</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.806</td>
<td>5.312</td>
<td>6.117</td>
<td>6.983</td>
<td>8.006</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.296</td>
<td>4.844</td>
<td>5.491</td>
<td>6.251</td>
<td>4.880</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.836</td>
<td>4.318</td>
<td>3.810</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.395</td>
<td>3.910</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.975</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

- The state price tree (via Duffie’s formula):

```
<table>
<thead>
<tr>
<th>Price</th>
<th>1.0000</th>
<th>.4877</th>
<th>.2369</th>
<th>.1146</th>
<th>.0552</th>
<th>.0264</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.4877</td>
<td>.4750</td>
<td>.3459</td>
<td>.3478</td>
<td>.2299</td>
<td>.1340</td>
</tr>
<tr>
<td></td>
<td>.2381</td>
<td>.3478</td>
<td>.3375</td>
<td>.2752</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.1165</td>
<td>.2269</td>
<td>.2752</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.0571</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>.0281</td>
</tr>
</tbody>
</table>
```

- The 6-month LIBOR tree:

```
<table>
<thead>
<tr>
<th>Rate</th>
<th>5.051</th>
<th>5.818</th>
<th>6.676</th>
<th>7.875</th>
<th>9.081</th>
<th>10.521</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.864</td>
<td>5.383</td>
<td>6.211</td>
<td>7.107</td>
<td>8.168</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.343</td>
<td>4.903</td>
<td>5.567</td>
<td>6.349</td>
<td>4.940</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.873</td>
<td>4.364</td>
<td>4.940</td>
<td>3.847</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.424</td>
<td>3.847</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.997</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```
3. Black-Derman-Toy Valuation of Caps (continued)

- Where did this stuff come from?
  - Short rate tree:
    * volatilities are (.125, .150, .165, .170, .175, .175)
      (taken from caplet volatilities above)
    * drift parameters chosen to match current spot rates
      (same as those listed above)
  - State prices: computed using Duffie’s formula
  - 6-month LIBOR:
    * one-period discount factor \( b \) related to short rate \( r \) by
      \[
      b = \exp(-rh/100) \iff r = -(100/h)\log b
      \]
    * 6-month LIBOR \( Y \) related to \( b \) by
      \[
      b = \frac{1}{1 + Yh/100} \iff Y = (100/h)(1/b - 1)
      \]
3. Black-Derman-Toy Valuation of Caps (continued)

- 2-year semianual interest rate cap at 5.5%
  - Cash flows are

\[
\begin{array}{cccccc}
\text{(na)} & 0.155 & 0.569 & 1.143 & (\text{na}) & (\text{na}) \\
0.000 & 0.000 & 0.345 & (\text{na}) & (\text{na}) \\
0.000 & 0.000 & (\text{na}) & (\text{na}) & (\text{na}) \\
0.000 & (\text{na}) & (\text{na}) & (\text{na}) & (\text{na})
\end{array}
\]

Comments:
* reminder: \((Y - K)^+\) paid one period later
* if we push payments back a period, they don’t fit into the tree (which node the following period?)
* solution: discount the payments and shift them back a period
* boxed node: payment is

\[
0.5 \frac{(Y - K)^+}{1 + Y/200} = 0.5 \frac{(6.211 - 5.500)^+}{1 + 6.211/200} = 0.345
\]

(think about this if it’s not clear)
* why only 3 periods? because we observe the final payment one period before it’s scheduled to be paid

- Value of option:
  * all-at-once method (multiply cash flows by state prices and add):

\[
\text{Cap Price} = 0.461
\]

(similar to our earlier answer)
4. Swaps

- A (plain-vanilla) interest rate swap is an agreement between two parties to exchange fixed and floating interest payments:

![Diagram of a swap agreement]

We say Counterparty 1 “pays fixed” and Counterparty 2 “receives fixed”

- Standard approach to valuation:
  - add principal to both sides
  - Counterparty 2 then has a long position in a bond and a short position in a floating rate note
  - bonds we value with discount factors and the FRN trades at par on reset dates

- Swap rates are par yields:
  - fixed payments of $S/m$ are worth
    \[
    (b^1_l + \cdots + b^\tau_l) \left(\frac{S}{m}\right) + b^\tau_l 100
    \]
    \[
    (\tau \text{ is the tenor of the swap})
    \]
  - FRN worth 100 at start
  - if we choose the swap rate $S$ to equate the initial values of the fixed and floating sides:
    \[
    S = m \times 100 \times \frac{1 - b^\tau_l}{\sum_j b^j_l}
    \]
4. Swaps

- Numerical examples (same data as before):

<table>
<thead>
<tr>
<th>Period ((j))</th>
<th>Disc Factor</th>
<th>Swap Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.975365</td>
<td>5.052</td>
</tr>
<tr>
<td>2</td>
<td>0.949999</td>
<td>5.194</td>
</tr>
<tr>
<td>3</td>
<td>0.924837</td>
<td>5.274</td>
</tr>
<tr>
<td>4</td>
<td>0.899541</td>
<td>5.358</td>
</tr>
<tr>
<td>5</td>
<td>0.874550</td>
<td>5.426</td>
</tr>
<tr>
<td>6</td>
<td>0.849939</td>
<td>5.483</td>
</tr>
</tbody>
</table>

Comments:

- swap rates are semi-annually compounded
- details for maturity = 4 periods (2 years):

\[
\sum_{j=1}^{4} b^j = 0.975365 + 0.949999 + 0.924837 + 0.899541
\]

\[
= 3.7497
\]

\[
1 - b^4 = 1 - 0.899541 = 0.10045
\]

\[
S = 200 \times \frac{0.10045}{3.7497} = 5.358
\]
5. Forward-Starting Swaps

- A **forward-starting swap** is an agreement to enter into a swap $n$ periods (say) in the future.

- Valuation follows a similar route:
  - fixed payments of $F/m$ are worth
    \[ (b_t^{n+1} + \cdots + b_t^{n+r}) \left( \frac{F}{m} \right) + b_t^{n+r}100 \]
  - FRN worth 100 at start, $b_t^n$ 100 now
  - the forward swap rate $F$ equates the values of the fixed and floating sides:
    \[ F = m \times 100 \times \frac{b_t^n - b_t^{n+r}}{\sum_j b_t^{n+j}} \]

- Example: 2-year swap starting in 1 year
  \[
  \sum_{j=3}^{6} b^j = 3.5489 \\
  b^2 - b^6 = 0.949999 - 0.849939 = 0.10006 \\
  F = 200 \times (0.10006/3.5489) = 5.639
  \]

  This is a little higher than either the 2- or 3-year swap rates, since it’s based on the “3 to 6” part of the forward rate curve.

- Forward-starting swaps are the underlying assets for common swaptions.
6. Swaptions

- Terminology for common swaptions
  - A *payer swaption* is an option to enter into a pay fixed swap: a call option on a pay fixed swap
  - A *receiver swaption* is an option to enter into a receive fixed swap: a call option on a receive fixed swap or a put option on a pay fixed swap
  - Typically European
  - A “1 into 5” is a one-year option to enter into a 5-year swap
  - Strike generally quoted as a rate
  - General notation: $n$ is the maturity of the option and $\tau$ is the tenor or term of the underlying swap

- Other structures
  - American or Bermudan call features
  - Extendible: the option to extend the maturity of an existing swap
  - Cancellable: the option to cancel an existing swap
6. Swaptions (continued)

- One view of swaptions (option on a bond):
  - a claim to a swap at rate \( K \) in \( n \) periods:
    \[ V_{t+n}(K)^+ \]
    where \( V_{t+n} \) is the value of the swap in \( n \) periods
  - requires only the ability to value fixed-rate bonds (the floating side trades at par)
  - we generally use this approach when we value swaptions in binomial models
6. Swaptions (continued)

- Another view of swaptions (option on the swap rate):
  - the owner of a payer swaption has a claim to the stream of equal payments
    \[ m^{-1}(S_{t+n} - K)^+ \]
    in periods \( t + n + 1, t + n + 2, \ldots, t + n + \tau \)
  - why?
    * an optional short position in a “rate-\(K\)” swap
    * ... is equivalent to a short position in a rate-\(K\) swap and a long position in a swap at the market rate \(S\) at time \(t + n\) (since the latter is priced to trade at zero)
    * each swap position has a fixed rate bond on one side and a floating rate note on the other
    * the floating rate notes cancel (one is long, the other short, and they have the same value)
    * ... leaving us with a short position in a rate-\(K\) bond and a long position in a rate-\(S_{t+n}\) bond
    * ... which generates cash flows of \(m^{-1}(S_{t+n} - K)^+\) at dates \(t + n + 1, \ldots, t + n + \tau\)
  - we use this approach with Black-Scholes valuation
7. Black-Scholes Valuation of Swaptions

- The Black-Scholes formula for a payer swaption is

\[
\text{Swaption Price } = m^{-1} \left[ BF\Phi(d) - BK\Phi(d - (nh)^{1/2}v) \right]
\]

\[
d = \frac{\log(F/K) + (nh)v^2/2}{(nh)^{1/2}v}
\]

\[
F = \text{forward-starting swap rate}
\]

\[
K = \text{strike rate}
\]

\[
m = \text{number of payments per year}
\]

\[
h = 1/m = \text{time between payments in years}
\]

\[
n = \text{maturity of swaption in length-}h \text{ periods}
\]

\[
nh = \text{maturity of swaption in years}
\]

\[
B = b_{t}^{n+1} + b_{t}^{n+2} + \cdots + b_{t}^{t+\tau}
\]

\[
\Phi = \text{cumulative normal distribution function}
\]

\[
v = \text{annualized volatility}
\]

Comments:

- Think of this as a call option on \( S \) with strike \( K \)
- \( B \) values the series of payments of \((S_{t+n} - K)^+\)
- Common variants express \( B \) in different ways
- By expressing \( F \) as a percentage we get the price per 100 notional
7. Black-Scholes Valuation of Swaptions (continued)

- Numerical example: 1-year option on 2-year swap
  - Recall from forward-starting swap: $F = 5.639$, $B = 3.5489$
  - Volatility (from quote sheet): $v = 15.55\%$
  - Swaption prices for various strikes

<table>
<thead>
<tr>
<th>Strike</th>
<th>Swaption Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.639</td>
<td>0.6201</td>
</tr>
<tr>
<td>5.750</td>
<td>0.5326</td>
</tr>
<tr>
<td>6.000</td>
<td>0.3698</td>
</tr>
<tr>
<td>7.000</td>
<td>0.0648</td>
</tr>
</tbody>
</table>

(Prices are in dollars per hundred notional)
8. Black-Derman-Toy Valuation of Swaptions

- Value a 1-year payer swaption on a 2-year swap

- Underlying: a forward-starting swap with maturity $n = 2$ and tenor $\tau = 4$ (both measured in half-years)
  - Regard f-s swap as short position in fixed rate bond and long position in floating rate note
  - Floating rate note trades for 100 in 2 periods, when the swap starts
  - Fixed rate bond has cash flows of $F/2 = 2.8195$ in all states in periods (3,4,5,6) plus principal of 100 in period 6:

\[
\begin{array}{cccc}
\text{(na)} & \text{(na)} & \text{(na)} & \text{(na)} \\
\text{(na)} & \text{(na)} & \text{(na)} & \text{(na)} \\
\end{array}
\begin{array}{cccc}
2.82 & 2.82 & 2.82 & 2.82 \uparrow \\
2.82 & 2.82 & 2.82 & 2.82 \uparrow \\
2.82 & 2.82 & 2.82 & 2.82 \uparrow \\
100.50 & 101.60 & 102.48 & 103.16 \downarrow \\
103.70 & 104.12 & 105.00 & 105.80 \downarrow \\
\end{array}
\]

Comments:

* last row: take cash flows from following period, and discount them back one period using the appropriate short rate; eg,

\[
104.12 = 2.82 + \exp(-2.975/200)(102.82)
\]

* earlier periods: fixed payments during the period of the swap
8. Black-Derman-Toy Valuation of Swaptions (continued)

- Valuation of underlying (continued)
  - Price path for fixed rate bond:

  \[
  \begin{array}{cccccc}
  95.00 & 95.97 & 97.39 & 99.44 & 99.48 & 100.50 \\
  98.83 & 100.13 & 101.85 & 102.85 & 101.60 & 101.60 \\
  102.35 & 103.79 & 103.16 & 103.70 & 102.48 & 102.48 \\
  105.34 & 104.04 & 104.98 & 104.12 & 103.16 & 103.16 \\
  \end{array}
  \]

- Swap includes short position in bond (above) plus long position in floating rate note (100 in period 2)

- Cash flows for swaption are therefore:

  \[
  \begin{array}{ccc}
  (\text{na}) & (\text{na}) & 2.608 \\
  (\text{na}) & 0.000 & 0.000 \\
  \end{array}
  \]

Comment:

\[ 2.608 = 100 - 97.392 \]

(long position in note, short position in bond)
8. Black-Derman-Toy Valuation of Swaptions (continued)

- Recursive swaption valuation:

```
    0.618 <--- 1.267 <--- 2.608
    0.000 <--- 0.000 <--- 0.000
```

Comment: the usual approach

- All-at-once valuation:

\[
0.618 = 0.2369 \times 2.608
\]

\(0.2369\) is the state price for node (2,2)

- Value (0.618) similar to Black-Scholes calculation (0.620)
Summary

1. Common options on fixed income instruments include caps (and floors) and swaptions.

2. Dealers often quote implied volatilities, which are based on Black-Scholes applied to the underlying rates.

3. Valuation is often done with binomial models, which are valued the same way we value any derivative instrument.

4. The Black-Derman-Toy model is based on lognormal rates, and in that sense is similar to applications of Black-Scholes.