You should feel free to speak to your colleagues about these problems — in fact, I encourage it — but write down for yourself any answers you submit. You are welcome to compute solutions to numerical examples if you find this easier.

1. (small open economy with adjustment costs) Consider a small open economy business cycle model characterized by the Bellman equation:

\[
J(k, a, z) = \max_{c, x, n} \log c + \gamma \log(1 - n) + \beta EJ(k', a', z')
\]

subject to

\[
k' = (1 - \delta)k + x - \kappa[(k' - k)/k]^2
\]
\[
a' = r(a + zk^\alpha n^{1-\alpha} - c - x)
\]
\[
z' = (1 - \varphi)\bar{z} + \varphi z + \sigma \varepsilon',
\]

where the $\varepsilon$'s are iid with mean zero and variance one.

(a) What are the first-order and envelope conditions?

(b) Let $\varepsilon = 0$ and $z = \bar{z}$. For a given value of $a$, use your answer to (a) to find steady state values of other variables.

(c) Compute a linear-quadratic approximation to the decision rules using these parameter values: $\beta = 0.99$, $r = 1/0.99$, $\gamma = 2$, $\alpha = 1/3$, $\delta = 0.1$, $\kappa = 0.8$, $\varphi = 0.95$, $\bar{z} = 1$, and $\sigma = 0.01$. What is the steady state? What are the eigenvalues of the controlled system?

(d) Describe the responses of consumption, investment, labor input, and output to a blip of size one to $z$.

(e) (extra credit) Some authors prefer the budget constraint,

\[
a' = ra + zk^\alpha n^{1-\alpha} - c - x.
\]

In what ways does the economic content differ from our version?

2. (bond pricing) Consider the basic model of bond pricing, consisting of these equations:

\[
-\log m_{t+1} = \delta + z_t + \lambda \varepsilon_{t+1}
\]
\[
z_{t+1} = (1 - \varphi)\mu + \varphi z_t + \sigma \varepsilon_{t+1},
\]

where $\{\varepsilon_t\} \sim NID(0, 1)$. The beauty of this model is that it delivers bond yields that are linear functions of the state variable $z$. If $b_t^n$ is the price of a claim to one (dollar,
say) in \( n \) periods, yields are defined by \( b^n_t = \exp(-ny^n_t) \), or \( y^n_t = -n^{-1} \log b^n_t \). The model delivers bond prices satisfying

\[
-\log b^n_t = A_n + B_n z_t
\]

for coefficients \((A_n, B_n)\) that depend on the model’s parameters \((\mu, \varphi, \sigma, \delta, \lambda)\). Bond prices follow from the pricing relation, \( E_t(m_{t+1}r_{t+1}) = 1 \), and “Ito’s lemma”: if \( \log x \sim N(a, b) \), then \( \log E(x) = a + b/2 \).

Our mission is to estimate the parameters of the model when the parameters are chosen so that \( z_t = y^1_t \) (take it from me that this is a normalization).

(a) Derive recursions that express \((A_{n+1}, B_{n+1})\) as functions of \((A_n, B_n)\) and the model’s parameters.

(b) What are the starting values \((A_0, B_0)\)? What value of \( \delta \) implies \( y^1_t = z_t \)? For extra credit: Why is this a normalization? (Think: How might I estimate \( \delta \)?)

(c) We’re going to use four moment conditions to “estimate” the four remaining parameters. (Think of this as poor man’s GMM.) They are: the one-year rate has mean 7.544\% (0.07544), standard deviation 2.672\%, and autocorrelation 0.970; and the mean 10-year yield is 8.529. Use the first three moments to estimate \((\mu, \varphi, \sigma)\). Use the fourth moment to estimate \( \lambda \). Hint: you will probably have to do this numerically.

(d) Use your estimates to compute the standard deviation of \( \log m \). How does it compare to annual estimates of Hansen-Jagannathan bounds?

(e) (extra credit) Describe the dynamics of \( \log m \).

3. (currency pricing, extra credit) We’ll use a similar model to think about prices of currencies. Suppose we have two countries, home and foreign, with bonds denominated in dollars and euros respectively. The dollar pricing kernel is characterized by

\[
-\log m^d_{t+1} = \delta + z_t + \lambda_1 \varepsilon_{t+1} + \lambda^* \varepsilon^*_t + 1 - \varphi \mu + \varphi z_t + \sigma \varepsilon_{t+1},
\]

where \( \{\varepsilon_t\} \) and \( \{\varepsilon^*_t\} \sim NID(0, 1) \). The euro pricing kernel is characterized by

\[
-\log m^e_{t+1} = \delta + z^*_t + \lambda^* \varepsilon_{t+1} + \lambda \varepsilon t + 1 - \varphi \mu + \varphi z^*_t + \sigma \varepsilon^*_{t+1},
\]

Exchange rates are implied by the relation: \( m^e_{t+1} = m^d_{t+1} (e_{t+1}/e_t) \), where \( e_t \) is the dollar price of one euro (the spot exchange rate) and \( e_{t+1}/e_t \) is the rate of depreciation of the dollar.

(a) Use methods similar to the previous problem to derive bond prices denominated in both currencies. In each case, the log of the bond price should be a linear function of the two state variables. Given the symmetry of the problem, doing this for one currency will suffice.

(b) Choose parameter values to match the four properties of bond prices plus the standard deviation of the depreciation rate, say 10% annually.
(c) With these parameter values, what is the correlation of $\log m$ and $\log m^*$?

(d) (double extra credit) What is the effect of adding noise $\{\eta_t\}$ to the log $m$ equation?

4. (portfolio choice) Consider portfolio choice in the nontraded goods economy: an exchange economy with $I$ countries, each represented by a single agent, and $I+1$ goods, one traded good and $I$ nontraded goods. Uncertainty is described by the usual event tree. In each state, the agent of country $i$ consumes $a_i$ units of the traded good and $b_i$ units of her own nontraded good. Her endowments are $w_i$ and $x_i$, respectively. Preferences are additive over time and across states, with period utility function

$$u(a, b) = \log a + \log b,$$

and agents have the same discount factor and probability assessments.

(a) Solve a Pareto problem for optimal allocations and implicit prices of traded and nontraded goods in each state.

(b) What is the price index $p$ that corresponds to aggregate consumption in an arbitrary country?

(c) Describe a set of Lucas trees that supports the optimum as a competitive equilibrium.

(d) What is the pricing kernel $m_i(s^t, s_{t+1})$ in country $i$, denominated in units of the local consumption composite?

(e) How are the pricing kernels in countries $i$ and $j$ related? What information does their relation contain about the effectiveness of risk-sharing across countries?