Easy EZ for DSGE

Harald Uhlig

1University of Chicago
Department of Economics
huhlig@uchicago.edu

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Outline

1. Introduction
2. The model
3. Analysis
   - Preparations
   - Parameters
4. Log-Linearization
   - Log-linearizing Epstein-Zin
   - The log-linear equations (no wedges)
5. Asset pricing
6. Conclusions
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Epstein-Zin preferences and DSGE models

- Epstein-Zin ("EZ") preferences: popular in asset pricing!
- Still challenging in DSGE models.
- Tallarini, Guvenen: numerical.
- Backus-Routledge-Zin.

“If the procedure is straightforward, the calculations are not. See Appendix B for the gruesome details. ... [Appendix B:] This is a complete mess; its essential feature for our purposes is that all of these coefficients are linear functions of the value function parameters.
Goal and Challenges

- Make the use of Epstein-Zin ("EZ") preferences (comparatively) easy in dynamic stochastic general equilibrium ("DSGE") models.
- Obtain insights.
- Compare, say, to Hansen-Heaton-Li, Tallarini, Guvenen.
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The model

- Representative agent with Epstein-Zin preferences.
- Allow for distortions ("wedges").
Preferences

\[ V_t = \left( (1 - \tilde{\beta}) (c_t \Phi(n_t))^{1-\rho} + \beta \mathcal{R}_t^{1-\rho} \right)^{\frac{1}{1-\rho}} \quad (1) \]

where \( \rho > 0, \rho \neq 1, 0 < \beta < 1, 0 < \tilde{\beta} < 1 \) and where

\[ \mathcal{R}_t = \left( E_t \left[ V_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}} \quad (2) \]

Note: labor matters!
The model

Constraints

- Budget constraint:
  \[(1 + \tau^c_t)c_t + (1 + \tau^x_t)x_t = (1 - \tau^w_t)w_t n_t + (1 - \tau^k_t)r_t k_t + s_t\] (3)

- Capital accumulation:
  \[k_{t+1} = \left(1 - \delta + h\left(\frac{x_t}{k_t}\right)\right) k_t\] (4)

where
  \[\tilde{\delta} = h(\tilde{\delta}), 1 = h'(\tilde{\delta}), \omega = -h''(\tilde{\delta})\tilde{\delta} > 0\] (5)
Production

- Production function:
  \[ y_t = f \left( \frac{A_t n_t}{k_t} \right) k_t \]  

- Wages and capital rental rates:
  \[ w_t = A_t f' \left( \frac{A_t n_t}{k_t} \right) \]
  \[ r_t k_t = y_t - w_t n_t \]
Closing the model

- Resources lost / Government spending:

\[ g_t = y_t - c_t - x_t \quad (7) \]

- Growth:

\[ \zeta_t = \frac{A_t}{A_{t-1}} \quad (8) \]

stationary. Mean: \( \bar{\zeta} \).

- wedges: stationary.
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Definitions

Define:

\[ \tilde{\delta} = \bar{\zeta} - 1 + \delta \]
\[ \tilde{\beta} = \beta \bar{\zeta}^{1-\rho} \]

Detrend:

\[ \bar{V}_t = \frac{V_t}{A_t}, \quad \bar{R}_t, \bar{c}_t, \bar{x}_t, \bar{k}_t, \bar{y}_t, \bar{\nu}_t, \bar{g}_t, \bar{s}_t \]
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Original parameters

- Basic parameters:
  \[ \bar{\zeta}, \beta, \delta, \rho, \gamma \]

- Wedges:
  \[ \bar{\tau}^c, \bar{\tau}^x, \bar{\tau}^w, \bar{\tau}^k, \bar{s} \]
Summarizing $h(\cdot), \Phi(\cdot), f(\cdot)$.

<table>
<thead>
<tr>
<th>abbrev.</th>
<th>expression</th>
<th>equality</th>
<th>sugg. value</th>
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<tbody>
<tr>
<td>$h(\tilde{\delta})$</td>
<td>per assumption:</td>
<td>$\tilde{\delta}$</td>
<td></td>
</tr>
<tr>
<td>$h'(\tilde{\delta})$</td>
<td>per assumption:</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$\omega = -h''(\tilde{\delta})\tilde{\delta}$</td>
<td>free: cap. adj. cost param.</td>
<td>$\in [0, 4\tilde{\delta}]$</td>
<td></td>
</tr>
<tr>
<td>$\theta = \frac{f'(\bar{n}/\bar{k})\bar{n}/\bar{k}}{f(\bar{n}/\bar{k})}$</td>
<td>$= \bar{w}\bar{n}/\bar{y}$ (labor share: obs.)</td>
<td>$2/3$</td>
<td></td>
</tr>
<tr>
<td>$1/\eta_D = \frac{-f''(\bar{n}/\bar{k})\bar{n}/\bar{k}}{f'(\bar{n}/\bar{k})}$</td>
<td>free: inv. lab. demand elast.</td>
<td>$1/3$</td>
<td></td>
</tr>
<tr>
<td>$\Phi(\bar{n})$</td>
<td>Normalization:</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$\nu = \frac{-\Phi'(\bar{n})\bar{n}}{\Phi(\bar{n})}$</td>
<td>$= \frac{1-\tau^w}{1+\tau^c} \theta \bar{y}/\bar{c}$ (observed)</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$1/\eta_S = \left(\frac{\Phi''(\bar{n})\bar{n}}{\Phi'(\bar{n})} - \nu\right)$</td>
<td>free: inv. labor supply elast.</td>
<td>$2$</td>
<td></td>
</tr>
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The value function

- **Original, detrended:**
  
  \[ \tilde{V}_t^{1-\rho} = (1 - \tilde{\beta}) (\tilde{c}_t \Phi(n_t))^{1-\rho} + \beta \tilde{R}_t^{1-\rho} \]
  
  \[ \tilde{R}_t = \left( E_t \left[ \tilde{\zeta}_{t+1} \tilde{V}_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}} \]

- **Log-linearized:**
  
  \[ \hat{V}_t = (1 - \hat{\beta}) (\hat{c}_t - \nu \hat{n}_t) + \hat{\beta} \hat{R}_t \]
  
  \[ \hat{R}_t = E_t \left[ \hat{\zeta}_{t+1} + \hat{V}_{t+1} \right] \]
The stochastic discount factor

- Original, detrended:

\[
M_{t+1} = \beta \zeta_{t+1}^\gamma \left( \frac{\hat{c}_{t+1}}{\hat{c}_t} \right)^{-\rho} \left( \frac{\Phi(n_{t+1})}{\Phi(n_t)} \right)^{1-\rho} \left( \frac{\tilde{V}_{t+1}}{\tilde{R}_t} \right)^{\rho-\gamma} \frac{1 + \tau_t^C}{1 + \tau_{t+1}^C}
\]

\[
1 = E_t [M_{t+1} R_{t+1}]
\]

- Log-linearized:

\[
\hat{M}_{t+1} = -\gamma \hat{\zeta}_{t+1} - \rho (\hat{c}_{t+1} - \hat{c}_t) - (1 - \rho) \nu (\hat{n}_{t+1} - \hat{n}_t)
\]

\[
+ (\rho - \gamma) (\hat{V}_{t+1} - \hat{R}_t) + \frac{1}{1 + \tau_t^C} (\hat{r}_t^c - \hat{r}_{t+1}^c)
\]

\[
0 = E_t [\hat{M}_{t+1} + \hat{R}_{t+1}]
\]
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Epstein-Zin

\[
\begin{align*}
\hat{V}_t &= \left(1 - \hat{\beta}\right) (\hat{c}_t - \nu \hat{n}_t) + \hat{\beta} \hat{R}_t \\
\hat{R}_t &= E_t \left[\hat{\zeta}_{t+1} + \hat{V}_{t+1}\right] \\
\hat{M}_{t+1} &= -\gamma \hat{\zeta}_{t+1} - \rho (\hat{c}_{t+1} - \hat{c}_t) - (1 - \rho) \nu (\hat{n}_{t+1} - \hat{n}_t) \\
&\quad + (\rho - \gamma) \left(\hat{V}_{t+1} - \hat{R}_t\right) \\
0 &= E_t \left[\hat{M}_{t+1} + \hat{R}_{t+1}^{(k)}\right]
\end{align*}
\]
\begin{align*}
\hat{n}_t &= \eta_S (\hat{w}_t - \hat{c}_t) \\
\hat{y}_t &= \theta \hat{n}_t + (1 - \theta) \hat{k}_t \\
\hat{k}_{t+1} &= (1 - \tilde{\delta}) \hat{k}_t + \tilde{\delta} \hat{x}_t - \hat{\zeta}_{t+1} \\
\hat{y}_t &= \bar{c} \hat{c}_t + \bar{x} \hat{x}_t \\
\hat{n}_t &= \hat{k}_t - \eta_D \hat{w}_t \\
\hat{r}_t &= -\frac{\theta}{1 - \theta} \hat{w}_t \\
\hat{R}^{(k)}_t &= -\hat{q}_{t-1} + \frac{\bar{r}}{R^{(k)}} \hat{r}_t + \frac{\zeta}{R^{(k)}} \hat{q}_t \\
\hat{q}_t &= \omega (\hat{x}_t - \hat{k}_t)
\end{align*}
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Surprises

- Introduce the **surprise operator**

\[
S_{t+k|t} = E_{t+k}[x] - E_t[x]
\]  

(9)

- Write \(S_{t+1}\) for \(S_{t+1|t}\). Thus,

\[
S_{t+1}[x_{t+1}] = x_{t+1} - E_t[x_{t+1}]
\]

- Iteration:

\[
S_{t+k|t} = S_{t+k} + S_{t+k-1} + \ldots + S_{t+1}
\]

(10)
Telescoping

Telescoping the log-linearized value function:

\[ \hat{V}_t + \hat{\zeta}_t = E_t \left[ \sum_{j=0}^{\infty} \tilde{\beta}^j \left( \left(1 - \tilde{\beta}\right) (\hat{c}_{t+j} - \nu \hat{n}_{t+j}) + \hat{\zeta}_{t+j} \right) \right] \]  

Therefore:

\[ \hat{M}_{t+1} = -\rho \left( \hat{\zeta}_{t+1} + \Delta \hat{c}_{t+1} \right) - (1 - \rho) \nu \Delta \hat{n}_{t+1} \]

\[ - (\gamma - \rho) S_{t+1} \left[ \sum_{j=0}^{\infty} \tilde{\beta}^j \left( \hat{\zeta}_{t+1+j} + \Delta \hat{c}_{t+1+j} - \nu \Delta \hat{n}_{t+1+j} \right) \right] \]
Multi-period discount factor: similar.

Use state-space representation of linearized model to write this in term of current innovations.

Can use this to calculate variances and risk premia ...

... which in turn can be used for steady-state corrections to the level of returns ...

which in turn imply an adjustment to steady state levels of capital etc..

Fixed point / loop. Schmitt-Grohe - Uribe.
Pricing Long-Term Risk:

- Consider: $\rho = 1$, i.e. $\log(c) + \log \Phi(n_t)$. Suppose $\Phi(n_t) \equiv \Phi$.
- Stochastic discount factor:
  \[
  \hat{M}_{t+1} = -\hat{\zeta}_{t+1} - \Delta \hat{c}_{t+1} \\
  - (\gamma - 1) S_{t+1} \left[ \sum_{j=0}^{\infty} \tilde{\beta}^j \left( \hat{\zeta}_{t+j+1} + \Delta \hat{c}_{t+j+1} \right) \right]
  \]
- Assume joint log-normality.
- The equity premium:
  \[
  \log E_t [R_{t+1}] - \log R_t^f = \Cov_t \left( \hat{R}_{t+1}, \hat{\zeta}_{t+1} + \Delta \hat{c}_{t+1} \right) \\
  + (\gamma - 1) \Cov_t \left( \hat{R}_{t+1}, \sum_{j=0}^{\infty} \tilde{\beta}^j \left( \hat{\zeta}_{t+1+j} + \Delta \hat{c}_{t+1+j} \right) \right)
  \]
Issues:

- Future fluctuations in consumption growth are hard to forecast...
- ... and may be uncorrelated with returns.
- What about leisure?
Autocorrelation of Consumption growth, $\Delta c$
Correlations: stock returns, future first-diff. log cons.
Correlations: stock returns, future first-diff. log leisure

Correlations $\text{corr}(r_t, \Delta l_{t+j})$
Autocorrelation of Leisure changes, $\Delta l$

Autocorrelations for $\Delta l_j$ (Quarters)
The equity premium with leisure

- The stochastic discount factor:

\[ \hat{M}_{t+1} = -\hat{\zeta}_{t+1} - \Delta \hat{c}_{t+1} \]

\[ - (\gamma - 1) S_{t+1} \left[ \sum_{j=0}^{\infty} \tilde{\beta}^j \left( \hat{\zeta}_{t+j+1} + \Delta \hat{c}_{t+j+1} - \nu \Delta \hat{n}_{t+j+1} \right) \right] \]

- The equity premium:

\[ \log E_t [R_{t+1}] - \log R_t^f = \text{Cov}_t \left( \hat{R}_{t+1}, \hat{\zeta}_{t+1} + \Delta \hat{c}_{t+1} \right) \]

\[ + (\gamma - 1) \text{Cov}_t \left( \hat{R}_{t+1}, \sum_{j=0}^{\infty} \tilde{\beta}^j \left( \hat{\zeta}_{t+1+j} + \Delta \hat{c}_{t+1+j} \right) \right) \]

\[ - (\gamma - 1) \nu \text{Cov}_t \left( \hat{R}_{t+1}, \sum_{j=0}^{\infty} \tilde{\beta}^j \Delta \hat{n}_{t+1+j} \right) \]
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- I have provided an easy method for including Epstein-Zin preferences in DSGE models.
- I have explored some resulting asset pricing implications.