The Epstein-Zin Stochastic Growth Model

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The goal is to work out loglinear decision rules for the stochastic growth model with Epstein-Zin/Kreps-Porteus preferences. Tallarini (JME 2000) is the benchmark, but we’d like something closer to an analytical solution.

**Epstein-Zin preferences**

The Epstein-Zin class of preferences can be summarized by

\[ U = V[c, \mu(U')] \]

where \( V \) (time aggregator) governs time preference and \( \mu \) (certainty equivalent) governs risk preference. The constant-elasticity Kreps-Porteus version consists of

\[ V(x, y) = [(1 - \beta)x^\rho + \beta y]\]^{1/\rho} \tag{1} \\
\[ \mu(x) = \left[E(x^\alpha)\right]^{1/\alpha} \tag{2} \]

with parameters \( \rho, \alpha < 1 \). If \( \alpha = \rho \) this reduces to time-additive preferences. We refer to (2) as a power certainty equivalent. Another popular choice is

\[ \mu(x) = -\alpha^{-1} \log E(e^{-\alpha x}) \],

with \( \alpha > 0 \), which we refer to as an exponential certainty equivalent.

Transformations. Preferences are invariant to increasing functions of utility, so we can express them (for example) as

\[ \tilde{U} = (1 - \beta)e^\rho + \beta \tilde{\mu}(\tilde{U}') \]
\[ \tilde{\mu}(x) = [E(x)^\gamma]^{1/\gamma} \]

with \( \tilde{U} = U^\rho \) and \( \gamma = \alpha/\rho \). If \( \gamma = 1 \), preferences take the traditional time-additive form. [If \( \rho < 0 \) we need to divide by \( \rho \) to keep the transformation increasing: ie, define \( \tilde{U} = U^\rho/\rho \). I skip that to keep the algebra a little simpler. If \( \rho = 0 \) we use the traditional log limit; see below.]

Log time preference. When \( \rho = 0 \) we use

\[ \log U = (1 - \beta) \log c + \beta \log \mu(U') \].

*Working notes. No guarantee of accuracy or sense.*
If we define \( \hat{U} = \log U \), then \( U = \exp(\hat{U}) \) and

\[
\hat{U} = (1 - \beta) \log c + \beta \log (EU^\alpha)^{1/\alpha} \\
= (1 - \beta) \log c + \beta \log \left( Ee^{\alpha \hat{U}} \right)^{1/\alpha} \\
= (1 - \beta) \log c + \beta \hat{\mu}(\hat{U}'),
\]

with

\[
\hat{\mu}(x) = \alpha^{-1} \log(Ee^{\alpha x}),
\]

an exponential certainty equivalent function in terms of transformed utility \( \hat{U} \). Comments:

(i) Tallarini (footnote 4) seems to be right: this has a power certainty equivalent function, with constant relative risk aversion, not constant absolute risk aversion. The derivation makes it clear that the log comes from time preference, but the \( \alpha \) comes from a power certainty equivalent. (ii) We still have the parameter restriction \( \alpha < 1 \), which seems weird in this log-exp setup (ie, shouldn’t \( \alpha \) be negative?). The explanation, of course, is that this isn’t exponential utility, it’s power utility transformed. (iii) What happens when \( \alpha = \rho = 0 \)? Use l’Hospital’s rule to show

\[
\hat{U} = (1 - \beta) \log c + \beta E(\hat{U}').
\]

(iv) How do we interpret time and risk preference in Anderson’s (JET 2005) preferences,

\[
U = (1 - \beta) c^\rho / \rho + \beta \mu(U')?
\]

Stan’s comment: Think of starting with the power certainty equivalent (2) and the time aggregator

\[
V(x, y) = [\exp x^\rho] y^\beta.
\]

Now do the same Tallarini-like transformation: define \( \hat{U} = \log(U) \) so that

\[
\log(U) = c^\rho + \beta \log \mu(U'),
\]

which implies

\[
\hat{U} = c^\rho + \beta \hat{\mu}(\hat{U}'),
\]

where

\[
\hat{\mu}(x) = \alpha^{-1} \log(E \exp[\alpha x])
\]

ie, CRRA in levels is equivalent to CARA in the logs. Note that the aggregator function looks odd, but has perfectly standard properties: Along deterministic paths

\[
MRS_{t+1,t} = \beta(c_{t+1}/c_t)^{\rho-1}
\]

which implies that \( 1/\beta - 1 \) is the constant marginal rate of time preference and \( 1/(1 - \rho) \) is the standard substitution elasticity.

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The log model

Here’s a popular textbook model whose solution is exactly loglinear. Current utility is log \( c \). Production is \( zk^\theta \), which is allocated between consumption and capital next period: \( zk^\theta = c + k' \). The shock follows \( \log z' = \varphi \log z + \varepsilon' \) with \( \{ \varepsilon \} \sim \text{NID}(0, \kappa_2) \).

Solution. We guess the value function takes the form

\[
J(k, z) = A + B \log k + C \log z,
\]

which implies a Bellman equation of the form

\[
A + B \log k + C \log z = \max_c \log c + \beta \mathbb{E} \left[ A + B \log (zk^\theta - c) + C(\varphi \log z + \varepsilon') \right].
\]

The foc and ecs are

\[
\begin{align*}
1/c &= \beta B / (zk^\theta - c) \\
B/k &= [\beta B / (zk^\theta - c)] \theta z k^{\theta - 1} \\
C/z &= [\beta B / (zk^\theta - c)] k^\theta + \beta C \varphi / z.
\end{align*}
\]

The decision rule is

\[
c = (1 - \beta \theta) zk^\theta,
\]

which is loglinear, and the parameters of the value function include

\[
B = \left( \frac{\theta}{1 - \beta \theta} \right), \quad C = \left( \frac{1}{(1 - \beta \theta)(1 - \beta \varphi)} \right).
\]

Note that the derivatives of the value function are loglinear, which is what our “Campbell” approximation for the stochastic growth model was based on.

Kreps-Porteus version. Note that uncertainty in this model has no impact on the decision rule. One way to make uncertainty matter is to use the log version of EZ/KP preferences — namely equations (3,4). We modify the Bellman equation like this,

\[
J(k, z) = \max_c \log c + \beta \mu [J(k', z')],
\]

where \( \mu \) is the exponential certainty equivalent function (4). With the loglinear value function (5), the certainty equivalent is

\[
\mu(J') = A + B \log k' + C' \varphi \log z + \alpha C^2 \kappa_2.
\]

Here risk affects the value function through the constant \( A \) but has no impact on the decision rule. In short, this isn’t the example to illustrate the impact of risk on the growth model.

[Comment. Why different from Tallarini? Because he doesn’t have a loglinear value function. He has quadratic approximations to the return function and the value function.]
Two-period growth model

Another simple example uses the general EZ/KP preferences (12) in a two-period stochastic growth model. Output each period is \( y = z k^\theta \) with \( z \) random. To keep this simple, we start with an initial period with \( z = 1 \) and arbitrary \( k \). In the second (and last) period, \( k' = k - c \) and we consume everything: \( c' = z'(k')^\theta \) with (say) \( \log z' \sim N(0, \kappa_2) \). What is the optimal decision? The maximization is

\[
\max_{c'} \left[ c^\rho + \beta \mu(c')^\rho \right] / \rho.
\]

What makes this relatively easy is that we can evaluate the certainty equivalent:

\[
\mu(c') = e^{\alpha \kappa_2/2} (k')^\theta = e^{\alpha \kappa_2/2} (k - c)^\theta
\]

[Note: \( k' \) is known in the initial period, so there’s no uncertainty to worry about.] The problem is then

\[
\max_c \left[ c^\rho + \beta e^{\alpha \kappa_2/2} (k - c)^\theta \rho \right] / \rho.
\]

Earlier we ignored the impact of risk (namely, \( \kappa_2 \)). Here it shows up as an adjustment to the discount factor: use \( \beta^* = \beta \exp(\rho \alpha \kappa_2/2) \). The foc is

\[
c^{\rho-1} = \beta^* \theta (k - c)^{\theta \rho - 1}.
\]

We’re looking for a decision rule \( \hat{c} = h_{\hat{k}} k \). Unless \( \rho = 0 \) (log utility) or 1 (portfolio choice) there’s no simple formula, but we can easily come up with an approximation. There’s no steady state in this problem, but we can imagine choosing a particular point to give us an approximation something like

\[
(\rho - 1) \hat{c} = (\theta \rho - 1)(k/k')\hat{k} - (c/k')\hat{c},
\]

or

\[
\hat{c} = \frac{(\theta \rho - 1)(k/k')\hat{k}}{(\rho - 1) + (\theta \rho - 1)(c/k')}.
\]

It has the same flavor as risk-sharing problems when agents have different risk aversion parameters. [Figure out \( c' \) etc??]

Conjecture: the infinite-horizon model has the same property, namely the impact of risk shows up in the discount factor.

Stochastic growth model

Consider the stationary stochastic growth model with preferences (34) and technology

\[
k_{t+1} = g(k_t, z_t) - c_t
\]

\[
\log z_{t+1} = \varphi \log z_t + \varepsilon_{t+1}
\]
with \( \{\varepsilon_t\} \sim N(0, \kappa_2) \). We’ll generalize this later. Specific functions: \( y = z k^\theta \), \( g(k, z) = (1 - \delta) k + z k^\theta \). Then: \( g_k(k, z) = (1 - \delta) + \theta (y/k) \).

The problem is characterized by the Bellman equation

\[
J(k, z) = \max_c [c^\rho + \beta \mu(J(k', z'))^\rho]^{1/\rho}.
\]

The tricky part of this is evaluating the certainty equivalent. [Question: with expected utility, you simply take the expectation of the derivative of utility. Can we do something like that here and express the solution in terms of derivatives of the value function?] Given what we know about loglinear approximations, let us guess that the value function has (approximately) the form

\[
J(k, z) = p_0 k^{p_k} z^{p_z}.
\]

Then

\[
J' = p_0 (k')^{p_k} z^{p_z} e^{p_z \varepsilon'}
\]

\[
\mu(J') = p_0 (k')^{p_k} z^{p_z} e^{\alpha (p_z)^2 \kappa_2/2}.
\]

The Bellman equation becomes

\[
J(k, z) = \max_c [c^\rho + \beta^* p_0 [g(k, z) - c]^{p_k p_z \varepsilon'}]^{1/\rho}.
\]

where \( \beta^* = \beta e^{\alpha (p_z)^2 \kappa_2/2} \). The foc and ec imply ...