1. An application of the logarithmic interest rate model.
   
   (a) The short rate tree is

   \[
   \begin{array}{ccccccc}
   4.413 & 5.198 & 6.007 & 6.747 & \\
   3.850 & 4.450 & 4.999 & 3.703 & \\
   3.297 & 2.743 & \\
   \end{array}
   \]

   (b) The price path for the cap (100 notional) is

   \[
   \begin{array}{ccccccc}
   & 2.494 & 3.531 & 1.008 & \\
   1.256 & 2.351 & 3.350 & 1.017 & \\
   0.242 & 0.499 & 0.000 & 0.000 & \\
   0.000 & 0.000 & 0.000 & 0.000 & \\
   0.000 & 0.000 & 0.000 & 0.000 & \\
   \end{array}
   \]

   (c) For a given date in the tree (a column), you’ll notice more price variation at the top of the tree than the bottom. In option terminology, this reflects the greater “delta” when the caps are in the money. In fixed income terminology, this might be expressed as greater interest sensitivity (DV01, for example). The general point is that caps, like other option products, exhibit different sensitivity in different parts of the tree. A simple linear approximation, as in risk measures like the DV01 or duration, doesn’t capture the complexity of these instruments.

2. The approach is the same as that used for callable bonds: we find the cash flows from exercise, then work our way back through the tree and decide at each node whether we are better off exercising or not.

   (a) The underlying bond has price path (cum coupon)

   \[
   \begin{array}{ccccccc}
   & 103.08 & 100.83 & 99.10 & 98.31 & 98.55 & 100.04 \\
   104.95 & 102.92 & 101.62 & 101.10 & 101.51 & \\
   105.88 & 104.17 & 103.06 & 103.49 & \\
   106.13 & 104.55 & 102.64 & 104.13 & \\
   105.67 & 104.39 & 104.61 & \\
   \end{array}
   \]
(b) We value the put option separately then add it to the bond above. Since the bond prices include the coupon, an owner of the bond has the right to sell it back ("put" it) to the issuer at any point for 103. The cash flows from immediate exercise are the difference between 103 and the price, if positive, zero otherwise. From these cash flows, we follow the American exercise procedure to find the price path of the option. Nodes with boxes around them are those in which the option is exercised:

Note: (i) The option isn’t exercised at any particular time: depends on the circumstances. For this reason and others, shortcuts like yield-to-call and yield-to-worst are meaningless. (ii) Like the cap, the sensitivity is greater at the top of the tree. If we add the put to the bond (sum the two trees above), we’d find that price sensitivity is lower at the top of the tree, since the option effectively puts a floor on the price. (iii) The putable bond is worth 104.35, from which we would ordinarily subtract 3 to eliminate the current coupon. Stated differently, the put adds 1.27 to the value of the bond, which is the appeal to issuers: they can raise more money or pay a lower coupon.

All of this, of course, depends on our model. If we chose a different model, or even different parameters, the answer would change — one of the annoying, but unavoidable, features of instruments with state-contingent cash flows.

(c) As a result of the put, the bond has a short duration at high rates (it’ll be put) and long duration at low rates (where it’s essentially the same as the underlying bond). One way to see this is to replicate it. The putable bond’s cash flows at each node in the tree can be reproduced with quantities \((x_1, x_2)\) of the underlying bond and a one-period zero (paying 100 in each subsequent state). The numbers in this example are

You can see that the replicating strategy at the top of the tree is mostly the one-period zero, at the bottom mostly the underlying bond.