Bond Arithmetic

0. Overview

- Zeros and coupon bonds
- Spot rates and yields
- Day count conventions
- Replication and arbitrage
- Forward rates
- Yields and returns
1. Why Are We Doing This?

- Explain nitty-gritty of bond price/yield calculations
- Remark: “The devil is in the details”
- Introduce principles of replication and arbitrage
2. Zeros or STRIPS

- A zero is a claim to $100 in $n$ periods (price = $p_n$)

\[
\begin{array}{c}
\text{Pay } p_n \\
\text{Get } $100
\end{array}
\]

\[
\begin{array}{c}
t \\
\hline
\text{ } \\
\hline
\text{Pay } p_n \\
\text{Get } $100
\end{array}
\]

- A spot rate is a yield on a zero:

\[
p_n = \frac{100}{(1 + y_n/2)^n}
\]

- US treasury conventions:
  - price quoted for principal of 100
  - time measured in half-years
  - semi-annual compounding
2. Zeros (continued)

- A *discount factor* is a price of a claim to one dollar:

\[
d_n = \frac{p_n}{100} = \frac{1}{(1 + y_n/2)^n}
\]

- Examples (US treasury STRIPS, May 1995)

<table>
<thead>
<tr>
<th>Maturity (Yrs)</th>
<th>Price ($)</th>
<th>Discount Factor</th>
<th>Spot Rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>97.09</td>
<td>0.9709</td>
<td>5.99</td>
</tr>
<tr>
<td>1.0</td>
<td>94.22</td>
<td>0.9422</td>
<td>6.05</td>
</tr>
<tr>
<td>1.5</td>
<td>91.39</td>
<td>0.9139</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>88.60</td>
<td>0.8860</td>
<td>6.15</td>
</tr>
</tbody>
</table>
3. Compounding Conventions

- A yield convention is an arbitrary set of rules for computing yields (like spot rates) from discount factors.

- US Treasuries use semiannual compounding:

\[ d_n = \frac{1}{(1 + y_n/2)^n} \]

with \( n \) measured in half-years.

- Other conventions with \( n \) measured in years:

\[ d_n = \begin{cases} 
(1 + y_n)^{-n} & \text{annual compounding} \\
(1 + y_n/k)^{-kn} & \text{“k” compounding} \\
e^{-ny_n} & \text{continuous compounding (} k \rightarrow \infty \text{)} \\
(1 + ny_n)^{-1} & \text{“simple interest”} \\
(1 - ny_n) & \text{“discount basis”} 
\end{cases} \]

- All of these formulas define rules for computing the yield \( y_n \) from the discount factor \( d_n \), but of course they’re all different and the choice among them is arbitrary. That’s one reason discount factors are easier to think about.
4. Coupon Bonds

- Coupon bonds are claims to fixed future payments \( (c_n, \text{say}) \)

- They’re collections of zeros and can be valued that way:

\[
\text{Price} = d_1 c_1 + d_2 c_2 + \cdots + d_n c_n
\]

\[
= \frac{c_1}{(1 + y_1/2)} + \frac{c_2}{(1 + y_2/2)^2} + \cdots + \frac{c_n}{(1 + y_n/2)^n}
\]

- Example: Two-year “8-1/2s”
  Four coupons remaining of 4.25 each

\[
\text{Price} = 0.9709 \times 4.25 + 0.9422 \times 4.25 + 0.9139 \times 4.25 + 0.8860 \times 104.25
\]

\[
= 104.38.
\]

- Two fundamental principles of asset pricing:
  - Replication: two ways to generate same cash flows
  - Arbitrage: equivalent cash flows should have same price
5. Spot Rates from Coupon Bonds

- We computed the price of a coupon bond from prices of zeros

- Now reverse the process with these coupon bonds:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Maturity (Yrs)</th>
<th>Coupon</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.5</td>
<td>8.00</td>
<td>100.97</td>
</tr>
<tr>
<td>B</td>
<td>1.0</td>
<td>6.00</td>
<td>99.96</td>
</tr>
</tbody>
</table>

- Compute discount factors “recursively”:

\[
100.97 = d_1 \times 104 \quad \Rightarrow \quad d_1 = 0.9709
\]

\[
99.96 = d_1 \times 3 + d_2 \times 103
\]

\[
= 0.9709 \times 3 + d_2 \times 103 \quad \Rightarrow \quad d_2 = 0.9422
\]

- Spot rates follow from discount factors:

\[
d_n = \frac{1}{(1 + y_n/2)^n}
\]
5. **Spot Rates from Coupon Bonds (continued)**

- Do zeros and coupon bonds imply the same discount factors and spot rates?

- Example: Suppose Bond B sells for 99.50, implying $d_2 = 0.9377$ (B seems cheap)
  
  - Replication of B’s cash flows with zeros:
    
    \[
    3 = x_1 \times 100 \quad \Rightarrow \quad x_1 = 0.03
    \]
    
    \[
    103 = x_2 \times 100 \quad \Rightarrow \quad x_2 = 1.03
    \]
  
  - Cost of replication is
    
    \[
    \text{Cost} = 0.03 \times 97.09 + 1.03 \times 94.22 = 99.96
    \]
  
  - Arbitrage strategy: buy B and sell its replication
    Riskfree profit is $99.96 - 99.50 = 0.46$

- **Proposition.** If (and only if) there are no arbitrage opportunities, then zeros and coupon bonds imply the same discount factors and spot rates.

- Presumption: markets are approximately “arbitrage-free”

- Practical considerations: bid/ask spreads, hard to short
5. **Spot Rates from Coupon Bonds** (continued)

- Replication continued
  - Replication of coupon bonds with zeros seems obvious
  - Less obvious (but no less useful) is replication of zeros with coupon bonds
  - Consider replication of 2-period zero with \( x_a \) units of A and \( x_b \) units of B:
    
    \[
    0 = x_a \times 104 + x_b \times 3 \\
    100 = x_a \times 0 + x_b \times 103
    \]

  Remark: we’ve equated the cash flows of the 2-period zero to those of the portfolio \((x_a, x_b)\) of A and B
  * Solution:
    - \( x_b = 0.9709 = 100/103 \): hold slightly less than one unit of B, since the final payment (103) is larger than the zero’s (100)
    - \( x_a = -0.0280 \): short enough of A to offset the first coupon of B
  * We can verify the zero’s price:
    
    \[
    \text{Cost} = -0.0280 \times 100.97 + 0.9709 \times 99.96 = 94.22.
    \]

- Remark: even if zeros didn’t exist, we could compute their prices and spot rates.
6. Yields on Coupon Bonds

- Spot rates apply to specific maturities

- The *yield-to-maturity* on a coupon bond satisfies

\[
\text{Price} = \frac{c_1}{(1 + y/2)} + \frac{c_2}{(1 + y/2)^2} + \cdots + \frac{c_n}{(1 + y/2)^n}
\]

- Example: Two-year 8-1/2s

\[
104.38 = \frac{4.25}{(1 + y/2)} + \frac{4.25}{(1 + y/2)^2} + \frac{4.25}{(1 + y/2)^3} + \frac{104.25}{(1 + y/2)^4}
\]

The yield is \( y = 6.15\% \).

- Comments:
  - Yield depends on the coupon
  - Computation: guess \( y \) until price is right
7. Par Yields

- We’ve found prices and yields for given coupons

- Find the coupon that delivers a price of 100 (par)

  \[
  \text{Price} = 100 = (d_1 + \cdots + d_n)\text{Coupon} + d_n100
  \]

  The annualized coupon rate is

  \[
  \text{Par Yield} = 2 \times \text{Coupon} = 2 \times \frac{1 - d_n}{d_1 + \cdots + d_n} \times 100
  \]

- This obscure calculation underlies the initial pricing of bonds and swaps (we’ll see it again)
8. Yield Curves

- A yield curve is a graph of yield $y_n$ against maturity $n$

- May 1995, from US Treasury STRIPS:
9. Estimating Bond Yields

- Standard practice is to estimate spot rates by fitting a smooth function of $n$ to spot rates or discount factors.

- “Noise”: bid/ask spread, stale quotes, liquidity (on/off-the-runs), coupons, special features.

- Reminder that the frictionless world of the proposition is an approximation.
10. Day Counts for US Treasuries

- **Overview**
  - Bonds typically have fractional first periods
  - You pay the quoted price plus a pro-rated share of the first coupon (accrued interest)
  - Day count conventions govern how prices are quoted and yields are computed

- **Details**
  - Invoice price calculation (what you pay):
    \[
    \text{Invoice Price} = \text{Quoted Price} + \frac{u}{u+v} \text{Coupon}
    \]
    \[
    \text{Accrued Interest} = \frac{u}{u+v} \text{Coupon}
    \]
    \[
    u = \text{Days Since Last Coupon}
    \]
    \[
    v = \text{Days Until Next Coupon}
    \]
  - Yield calculation ("street convention"):  
    \[
    \text{Invoice Price} = \frac{\text{Coupon}}{(1 + y/2)^w} + \frac{\text{Coupon}}{(1 + y/2)^{w+1}} + \cdots + \frac{\text{Coupon} + 100}{(1 + y/2)^{w+n-1}}
    \]
    \[
    w = \frac{v}{u+v}
    \]
    \[
    n = \text{Number of Coupons Remaining}
10. Day Counts for US Treasuries (continued)

- US Treasuries use “actual/actual” day counts for $u$ and $v$ (ie, we actually count up the days)

- Example: 8-1/2s of April 97 (as of May 95)

<table>
<thead>
<tr>
<th>Issued</th>
<th>April 16, 1990</th>
</tr>
</thead>
<tbody>
<tr>
<td>Settlement</td>
<td>May 18, 1995</td>
</tr>
<tr>
<td>Matures</td>
<td>April 15, 1997</td>
</tr>
<tr>
<td>Coupon Freq</td>
<td>Semiannual</td>
</tr>
<tr>
<td>Coupon Dates</td>
<td>15th of Apr and Oct</td>
</tr>
<tr>
<td>Coupon Rate</td>
<td>8.50</td>
</tr>
<tr>
<td>Coupon</td>
<td>4.25</td>
</tr>
<tr>
<td>Quoted Price</td>
<td>104.19</td>
</tr>
</tbody>
</table>

Time line:

<table>
<thead>
<tr>
<th>Previous Coupon</th>
<th>Settlement Date</th>
<th>Next Coupon</th>
</tr>
</thead>
<tbody>
<tr>
<td>4/15/95</td>
<td>5/18/95</td>
<td>10/15/95</td>
</tr>
</tbody>
</table>

$u =$
$v =$
n =$
10. Day Counts for US Treasuries (continued)

Price calculations:

\[
\begin{align*}
\text{Accrued Interest} & = \\
\text{Invoice Price} & = 
\end{align*}
\]

Yield calculations:

\[
\begin{align*}
w & = \\
d & = 1/(1 + y/2) \quad \text{(to save typing)} \\
104.95 & = d^w (1 + d + d^2 + d^3) 4.25 + d^{w+3}100 \\
\Rightarrow & \quad d = 0.97021 \\
y & = 6.14\%
\end{align*}
\]

(The last step is easier with a computer)
11. Other Day Count Conventions

- US Corporate bonds (30/360 day count convention)
  (roughly: count days as if every month had 30 days)

Example: Citicorp’s 7 1/8s

Settlement: June 16, 1995
Matures: March 15, 2004
Coupon Freq: Semiannual
Quoted Price: 101.255

Calculations:

\[ n = \]
\[ u = \]
\[ v = \]
\[ w = \]

Accrued Interest =
Invoice Price =

\[
\text{Invoice Price} = d^w \left( \frac{1 - d^n}{1 - d} \right) \text{ Coupon} + d^{w+n-1}100
\]

\[ \Rightarrow \quad y = 6.929\% \]

Remark: the formula works, don’t sweat the details!
11. Other Day Count Conventions (continued)

- Eurobonds (30E/360 day count convention)
  (ie, count days as if every month has 30 days)

Example: IBRD 9s, dollar-denominated

<table>
<thead>
<tr>
<th>Settlement</th>
<th>June 20, 1995</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matures</td>
<td>August 12, 1997</td>
</tr>
<tr>
<td>Coupon Freq</td>
<td>Annual</td>
</tr>
<tr>
<td>Quoted Price</td>
<td>106.188</td>
</tr>
</tbody>
</table>

Calculations:

\[ n = \]
\[ u = \]
\[ v = \]
\[ w = \]

Accrued Interest

Invoice Price

\[ \text{Invoice Price} = d^w \left( \frac{1 - d^n}{1 - d} \right) \text{ Coupon} + d^{w+n-1} \times 100 \]

\[ \Rightarrow y = 5.831\% \]

Remark: the formula works for all coupon frequencies with the appropriate modification of \( d \) [here \( d = 1/(1 + y) \)]
11. Other Day Count Conventions (continued)

- Flow chart for computing bond yields
  - Determine: Coupon Rate and Coupon Frequency 
    \((k\text{ per year, say})\)
  - Compute:
    \[
    \text{Coupon} = \text{Coupon Rate}/k
    \]
  - Compute Accrued Interest, Invoice Price, and \(w\) using appropriate day-count convention
  - Computing the yield
    * Define
    \[
    d = \frac{1}{1 + y/k}
    \]
    * Find the value of \(d\) that satisfies
    \[
    \text{Invoice Price} = d^w \left(\frac{1 - d^n}{1 - d}\right) \text{Coupon} + d^{w+n-1}100
    \]
    * Compute \(y\) from \(d\):
    \[
    y = k(1/d - 1)
    \]
11. Other Day Count Conventions (continued)

- Eurocurrency deposits
  (generally actual/360 day count convention)

Example: 6-month dollar deposit in interbank market

<table>
<thead>
<tr>
<th>Settlement</th>
<th>June 22, 1995</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matures</td>
<td>December 22, 1995</td>
</tr>
<tr>
<td>Rate (LIBOR)</td>
<td>5.9375</td>
</tr>
</tbody>
</table>

Cash flows: pay (say) 100, get 100 plus interest

Interest computed by

\[
\text{Interest} = \text{LIBOR} \times \frac{\text{Actual Days to Payment}}{360}
\]

\[
= 5.9375 \times \left(\frac{183}{360}\right) = 3.018.
\]

Remarks

- Can be denominated in any currency
- “Interest” is analogous to \( y/2 \) in treasury formulas
12. Forward Rates

- A one-period forward rate \( f_n \) at date \( t \) is the rate paid on a one-period investment arranged at \( t \) (“trade date”) and made at \( t + n \) (“settlement date”)

\[
\begin{array}{cccc}
\text{trade} & \text{settlement} & \text{maturity} \\
 t & t + n & t + n + 1 \\
\end{array}
\]

- Representative cash flows (scale is arbitrary)

\[
\begin{array}{cccc}
 t & t + n & t + n + 1 \\
 0 & -F & 100 \\
\end{array}
\]

with \( F = 100/(1 + f_n/2) \) (verify that rate is \( f_n \))

- Replication with zeros

\[
\begin{array}{cccc}
 t & t + n & t + n + 1 \\
 -p_{n+1} & xp_n & 100 \\
\end{array}
\]

Choose \( x \) to replicate cash flows of forward contract

\[
0 = -p_{n+1} + xp_n
\]

\[
\Rightarrow 1 + f_n/2 = \frac{p_n}{p_{n+1}} = \frac{d_n}{d_{n+1}}
\]
12. Forward Rates (continued)

- Sample forward rate calculations:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Price ($)</th>
<th>Spot Rate (%)</th>
<th>Forward Rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>97.09</td>
<td>5.99</td>
<td>5.99</td>
</tr>
<tr>
<td>1.0</td>
<td>94.22</td>
<td>6.05</td>
<td>6.10</td>
</tr>
<tr>
<td>1.5</td>
<td>91.39</td>
<td>6.10</td>
<td>6.20</td>
</tr>
<tr>
<td>2.0</td>
<td>88.60</td>
<td>6.15</td>
<td>6.29</td>
</tr>
</tbody>
</table>

- Forward rates are the marginal cost of one more period:

\[
\begin{array}{c|c|c|c}
\hline
y_3 & y_3 & y_3 \\
\hline
f_0 & f_1 & f_2 \\
\hline
\end{array}
\]

- Spot rates are (approximately) averages:

\[
y_n \approx n^{-1}(f_0 + f_1 + \cdots + f_{n-1}) \\
\approx n^{-1} \sum_{j=1}^{n} f_{j-1}
\]

(This is exact with continuous compounding.)
12. Forward Rates (continued)

- Forward and spot rates in May 1995:
13. Yields and Returns on Zeros

- Example: Six-month investments in two zeros

<table>
<thead>
<tr>
<th>Zero</th>
<th>Maturity</th>
<th>Price</th>
<th>Spot Rate (%)</th>
<th>Forward Rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.5</td>
<td>97.56</td>
<td>5.00</td>
<td>5.00</td>
</tr>
<tr>
<td>B</td>
<td>1.0</td>
<td>94.26</td>
<td>6.00</td>
<td>7.00</td>
</tr>
</tbody>
</table>

- Scenarios for spot rates in six months

<table>
<thead>
<tr>
<th>Spot Rates (%)</th>
<th>Scenario 1</th>
<th>Scenario 2</th>
<th>Scenario 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>8.00</td>
<td>5.00</td>
<td>2.00</td>
</tr>
<tr>
<td>$y_2$</td>
<td>9.00</td>
<td>6.00</td>
<td>3.00</td>
</tr>
</tbody>
</table>

“up 3”    “no change”    “down 3”

- One-period returns on zeros

\[ 1 + h/2 = \frac{\text{Sale Price}}{\text{Purchase Price}} \]

($h$ for holding period, which is six months here)

- Scenario 2 returns

\[ \begin{align*}
(A) \quad 1 + h/2 &= \frac{100}{97.56} = 1.025 \quad \Rightarrow \quad h = 0.0500 \\
(B) \quad 1 + h/2 &= \frac{97.56}{94.26} = 1.035 \quad \Rightarrow \quad h = 0.0700
\]
13. Yields and Returns on Zeros (continued)

- Six-month returns ($h$):

<table>
<thead>
<tr>
<th>Scenario 1</th>
<th>Return on A (%)</th>
<th>Return on B (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 1</td>
<td>5.00</td>
<td>4.02</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>5.00</td>
<td>7.00</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>5.00</td>
<td>10.08</td>
</tr>
</tbody>
</table>

- Remarks
  - Return on A is the same in all scenarios
  - Standard result when holding period equals maturity:
    \[(1 + h/2)^n = \frac{100}{p_n} = (1 + y/2)^n\]
  - Return on B depends on interest rate movements
14. Yields and Returns on Coupon Bonds

- One-period returns

\[ 1 + \frac{h}{2} = \frac{\text{Sale Price} + \text{Coupon}}{\text{Purchase Price}} \]

(buy and sell just after coupon payment)

- Return when held to maturity
  Needed: return \( r \) on reinvested coupons

  - Three-period example

\[
\begin{array}{cccc}
| & C' & C' & 100 + C' \\
| \hline
 t & t + 1 & t + 2 & t + 3 \\
\end{array}
\]

\[
(1 + h/2)^3 = \frac{(1 + r)^2C + (1 + r)C' + C' + 100}{\text{Purchase Price}}
\]

- Return depends on reinvestment rate \( r \) (arbitrary)
- No simple connection between return and yield

- Bottom line: yields are not returns
Summary

- Bond prices and discount factors represent the time-value of money

- Spot rates do, too

- Conventions govern the calculation of spot rates from discount factors

- Yields on coupon bonds are a common way of representing prices, but are not otherwise very useful

- Cash flows of coupon bonds can be “replicated” with zeros, and vice versa

- Replication and arbitrage relations apply to frictionless markets, but hold only approximately in practice

- Yields and returns aren’t the same thing