Consumption and Real Exchange Rates: A Correlation Puzzle

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Abstract

This paper links real exchange rate movements to fundamental variables. Standard international business cycle models predict a perfect correlation between ratios of domestic to foreign aggregate consumption and bilateral real exchange rates. However, this prediction is inconsistent with the empirical evidence. I use a dynamic exchange economy to show that allowing for default in securities markets in conjunction with trade frictions can account for this phenomenon.

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1 Introduction

Unravelling of the Bretton-Woods system of fixed exchange rates in 1972 generated demand for a structural model of exchange rate determination. One challenge to building such a structural model has been the exchange rate disconnect puzzle. Many studies have shown that exchange rate movements are unrelated to fluctuations in virtually any kind of fundamental variable, including GDP, money supply, interest rate or consumption. In particular, Backus and Smith (1993) show that a feature of international business cycle models with complete markets is a perfect correlation between the ratio of domestic to foreign aggregate consumption and the bilateral real exchange rate. Backus and Smith further show that this prediction is in sharp contrast to the empirical evidence. They find that the cross correlation between growth rates of bilateral consumption ratios and real exchange rates has a mean of 0.045 and a range of [-0.08,0.17]. In this paper I address the real exchange rate disconnect puzzle as a first step in the direction of building an empirically sound structural model. I argue that trade and asset market frictions are some of the key ingredients necessary for resolving this puzzle.

To break the tight theoretical relation between consumption and exchange rate, frictions need to be introduced. Trade frictions, nominal rigidities and asset market frictions are the three types of frictions most relevant for the exchange rate disconnect problem. Previous attempts at resolving the discrepancy between theory and empirical evidence using structural models have focused on two of these frictions. However, nominal rigidities and trade frictions have been unsuccessful in reconciling theory with the empirical evidence.\(^1\) In this paper I argue that it is the interaction of goods trade and asset market frictions that holds the most promise in resolving the exchange rate disconnect puzzle. I introduce limited commitment in asset markets and obtain theoretical predictions for the correlation of real exchange rate and relative consumption on the order of -0.4, a significant improvement on the body of work currently available. Limited commitment in asset markets, discussed later in greater detail, augments a complete markets framework with an option to default on assets. In order to keep trade in assets going an agent with a low income shock needs to be borrowing constrained in the present and future periods in order to eliminate unsustainable debt levels. This introduces an extra, time varying, component into the relation between the consumption ratio and real exchange rate that breaks the perfect correlation of the complete markets model.

Existing models of exchange rate behavior fall into two broad classes. One class of models uses Arbitrage Pricing Theory (APT) to derive parity conditions that determine exchange rates. This class of models postulate exchange rates as functions of 'fundamental variables', such as relative money supplies, interest rates and outputs across countries, in an ad hoc manner. The ad hoc nature of the functional forms would have been less relevant if it generated serious gains in predictive capacity, but this class of models is generally outperformed by a

\(^1\)For example Chari, Kehoe and McGrattan (2002), Betts and Devereaux (2000) and Dumas (1992)
naive random walk, limiting their effectiveness\textsuperscript{2}. Another class of models takes a more structural approach to the problem by relating real exchange rates to ratios of marginal utilities of representative consumers across countries. A standard reference is Backus and Smith (1993). However, an empirical assessment of stripped down versions of this class of structural models yields much room for improvement\textsuperscript{3}. Even richer versions, such as Betts and Devereux (2000) or Chari, Kehoe and McGrattan (2002), (CKM hereafter), that incorporate nominal rigidities and monetary shocks remain limited in their effectiveness. CKM is particularly enlightening since they rigorously examine monetary models with flexible and sticky prices under the various hypotheses popular in the literature. CKM further shows that the Backus and Smith result of a structural link between relative consumptions and bilateral real exchange rates applies in monetary models as well and generates theoretical predictions that are the most at odds with the data.

The finding that limited commitment in asset markets breaks the tight link between consumption and real exchange rate is in line with a nascent literature pointing to asset market frictions as a promising tool in modeling exchange rate behavior. Cochrane, Brandt and Santa-Clara (2005) identify asset markets as the primary drivers of the disconnect between consumption and real exchange rates. Sarkissian (2003) shows that incomplete consumption risk sharing can generate realistic time-varying risk premiums in the foreign exchange market. Choi (2005) identifies an empirical relationship linking exchange rates and consumption ratios by introducing relative trade flows into the analysis. Furthermore, her results clearly point to a model with either taste shocks or limited commitment as potentially giving rise to that empirically consistent relationship.

The rest of the paper is arranged as follows: Section 2 describes the problem in detail and presents the empirical evidence, section 3 presents the benchmark complete markets and the limited commitment models, section 4 discusses the mechanics of the models, section 5 outlines a calibration of the model and section 6 concludes.

\section{Evidence}

Here I will set out the problem that the paper addresses by reviewing the related literature and documenting the empirical evidence on bilateral real exchange rates and fundamentals. In addition I present empirical evidence for a larger set of international macroeconomic data that helps evaluate model performance along several other important dimensions. The analysis uses annual data from the Organization for Economic Cooperation and Development (OECD) and World Bank databases covering the period 1980-2000 for Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Iceland, Italy, 

\textsuperscript{2}See Frankel and Rose (1996) for a thorough survey

\textsuperscript{3}For example Kollman (1995) or Head, Mattina and Smith (2003)
Japan, Korea, the Netherlands, New Zealand, Norway, Portugal, Spain, Sweden, the United Kingdom and the United States.\footnote{The sample of countries is chosen based on the availability of sectoral consumption and GDP data. For a detailed description of the series see the data appendix}

\section*{2.1 Data}

The real exchange rate is based on the purchasing power parity and is a key measure of competitiveness for an open economy. It is defined as

\[ RER = \frac{eP^*}{P} \]

where \( e \) is the direct quote nominal exchange rate, \( P^* \) is the price level in the foreign country and \( P \) is the domestic price level. Common measures of price levels are the consumer price index, the wholesale price index or the consumption price deflator. The consumption price deflator lends itself naturally as a measure of price level when examining the relationship between real exchange rates and relative consumptions.

The empirical evidence on real exchange rate fluctuations is quite stark. Figure 1 shows the first difference of natural logarithms of bilateral real exchange rate and the ratio of consumptions for the United States and Japan. While not a statistical measure it illustrates the apparent lack of any relationship between relative real consumption and real exchange rate, with brief periods of co-movement followed by equally brief periods of counter-movement. Table 1 documents the correlation between level and log-linear filtered U.S. bilateral real exchange rate and relative consumptions for the twenty industrialized countries in the sample. These two transformations are commonly used in the literature and they show similar results. The numbers for the correlation between real exchange rate and relative real consumption are consistent with various previous studies, including Backus and Smith (1993) and Corsetti, Dedola and Leduc (2004). The correlation is weak, with a mean of -0.09 and 0.05 and standard deviations of 0.4 and 0.3 for level and linear filter, respectively. These low correlations illustrate the central issue of an apparently weak connection between fundamentals and real exchange rates.

Real exchange rate movements are characterized by large and persistent deviations from purchasing power parity (PPP). Under standard theory, where PPP holds, nominal exchange rate movements should directly offset relative price movements, resulting in unitary real exchange rates. However in practice nominal exchange rate movements and relative price movements are not synchronized in the short run leading to large deviations from unity in the real exchange rate. In low inflation, industrialized countries, that are the focus of this study, most of the volatility in real exchange rates comes from nominal exchange rate movements. In emerging market countries, where two- and three-digit inflation rates are common, nominal exchange rate movements would make up only a modest fraction of real exchange rate fluctuations. Formally the deviations from PPP are summarized by examining the half life of innovations to
real exchange rates which are obtained from the following regression equation

\[ \log (RER_t) = \beta_0 + \beta_1 t + \beta_2 \log (RER_{t-1}) + \epsilon_t \]

where \( RER \) is the real exchange rate defined using the consumption price deflator as price level and \( \epsilon \) is white noise. Constructing the twenty possible bilateral U.S. real exchange rates in the sample the values of \( \beta_2 \) range from 0.607 for the Netherlands implying a half life of 17 months to 0.882 for Canada implying a half life of 66 months. The full set of results is presented in table 2.

A model of real exchange rate and fundamental fluctuations will also generate theoretical predictions for a larger set of economic variables. These model predictions can be compared with the empirical evidence for a larger set of international data in assessing the quality of the model. An economy is summarized by its national income and product accounts (NIPA). The NIPA break down gross domestic product into private and government consumption, investment and the trade balance. In the model presented in section 3 there is no government or investment, thus data on these items in the NIPA are omitted. Correlations with the U.S. variable for linearly detrended aggregate and disaggregated consumption and gross domestic product are presented in table 3. The average correlation of aggregate output for the twenty countries is 0.837. The average correlation of aggregate consumption is 0.176. At the disaggregated level the average correlation with the United States is 0.768 for tradeable gross domestic product and 0.796 for nontradeable gdp. Disaggregated consumption correlations are 0.128 for the traded and 0.078 for the nontraded sectors. Output correlation across countries that is higher than the correlation of consumption at both aggregate and disaggregated levels is a feature of the data that presents a significant challenge for structural models with risk sharing.

2.2 Literature

The relatively large literature linking fundamentals to exchange rates has by and large produced negative empirical results, rejecting existing theoretical models. At the same time a number of studies point to preference shocks as a device for reconciling theory with empirical evidence.\(^5\) However, preference shocks are an unsatisfactory device for a structural theory of exchange rate fluctuation. One of the contributions of this paper is to show that introducing default in asset markets is observationally equivalent to preference shocks in a two sector economy. This is an important result because asset market frictions, unlike preference shocks, can be measurable and predictable. The related literature is surveyed in order to place the current study in a broader context.

Meese and Rogoff (1983) challenged the validity of the class of models most used to forecast exchange rates in their empirical analysis. This class of models, working under the assumption that the exchange rate is the relative price of foreign and domestic money, attempt to tie the exchange rate to money supplies using arbitrage pricing theory. Starting with a money market equilibrium

\(^5\) For example Stockman and Tesar (1995) or Kollman (1995)
equation, \( m_t - p_t = \beta y_t + \alpha i_t + \epsilon_t \), for home and foreign countries these models derive the expression for the exchange rate

\[
p_t - p_t^* = (m_t - m_t^*) - \beta (y_t - y_t^*) + \alpha (i_t - i_t^*) - (\epsilon_t - \epsilon_t^*)
\]

This equation assumes that the uncovered interest parity (UIP) holds even though numerous studies have documented large departures from UIP in international financial data. Further identification of the exchange rate relies either on the PPP or a price friction adjusted version of it that attempts to capture the observed large departures from PPP in price and exchange rate data. This class of exchange rate models are based on the assumption of full arbitrage in financial markets. One of the drawbacks of such models is that they only implicitly incorporate individual behavior that ultimately drives all the results. Distortions, such as asset market or information frictions, or transaction costs, to name a few, could significantly alter underlying individual behavior but would not be reflected in this class of models. The main results obtained by Meese and Rogoff (1983) and confirmed ten years later by Frankel and Rose (1996) show that a naive random walk predicts exchange rates as well as this class of models. The ad hoc nature of the theoretical relationships is one of the key weaknesses of this class of models, making them inflexible to different aspects of individual behavior.

A more structural approach to exchange rate determination was taken by Backus and Smith (1993), (B&S hereafter). In their model, used as the benchmark in this paper, key aspects of individual behavior lead to a straightforward relationship between real exchange rate and relative consumption. B&S model each country as a utility maximizing agent with preferences over consumption of multiple goods. In equilibrium marginal utilities are equalized across countries leading to a direct theoretical link between the cross country ratio of consumptions and their bilateral real exchange rate. B&S further show that under complete markets the real exchange rate is related to relative consumption via

\[
RER_t = \lambda \left( \frac{C_t}{C_t^*} \right)^\gamma
\]

where \( RER \) is the real exchange rate, \( \lambda \) is a constant, \( C_t \) is the home country aggregate consumption at time \( t \) and \( * \) denotes the foreign variable. This relationship holds in a number of environments, including trade frictions and nontraded goods if markets are complete and there are no preference shocks\(^6\). The relationship between consumption and real exchange rate is intuitively appealing since choices between traded and nontraded goods or foreign and domestic consumptions should depend on the relative price – the real exchange rate. Clearly equation (1) predicts a perfect correlation between relative consumption and the real exchange rate in contrast to empirical evidence. The strength of the B&S model is that various complications and frictions can be introduced with clear implications for individual behavior affecting the link between the real exchange rate and relative consumption.

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Head, Mattina and Smith (2003), (HMS hereafter), use data for Canada, Denmark, Finland, France, Italy, Japan, New Zealand, Sweden, the U.K. and the U.S. to empirically test the relation between real exchange rate and relative consumption in a wide range of permutations on the B&S model. HMS introduce money, habit persistence, asset market frictions and alternative specifications for aggregate consumption in testing structural models of real exchange rate fluctuations. The results they find are negative, in the sense that the models do not perform well empirically. The specifications that most improve the empirical performance are asset market frictions and preference shocks. When HMS introduce asset market frictions by restricting the set of available securities to a single, uncontingent bond the empirical performance of the model improves but is still weak. Habit persistence in agent preferences also improves performance but, once again, the model remains weak empirically. The results are in line with the literature pointing to taste shocks or asset market frictions as a way of reconciling this class of structural models with the empirical evidence.\footnote{See Kollman(1995), Stockman and Tesar (1995)}

Two sophisticated analyses of the structural model are offered in Stockman and Tesar (1995) and Chari, Kehoe and McGrattan (2002). Stockman and Tesar (1995), ST hereafter, calibrate a two country real business cycle model with traded and nontraded sectors and compare the model predictions with the empirical evidence. Complete asset markets in the ST model result in a perfect correlation of relative consumption and real exchange rate, a result consistent with B&S. Significantly, the model is successful at matching second moments for most fundamental variables. ST further find that introducing taste shocks significantly improves model performance in matching the correlation of real exchange rate and relative consumption without detracting from the performance in matching second moments in fundamental variables. Chari, Kehoe and McGrattan, (CKM hereafter), calibrate a two country general equilibrium model with production, nominal rigidities and monetary shocks. They model the set of commodities as two traded goods, each exclusively produced by one of the countries. Sticky prices and monetary shocks generate deviations from PPP but do not affect the correlation of real exchange rate with relative consumption. CKM experiment with a number of environments but find that the real exchange rate is perfectly correlated with relative consumption under both complete and incomplete markets. The asset market friction CKM introduce is to limit the set of traded securities to a single uncontingent bond. The only improvement in model performance in replicating the correlation of relative consumption and real exchange rate is generated by making preferences non-separable in consumption and leisure. This specification generates effects similar to preference shocks by changing marginal utility through variation in the labor input in response to productivity and monetary shocks. The calibrated models ST and CKM examine lead to the conclusion that demand shocks or asset market frictions are necessary for reconciling empirical evidence and theory on the correlation of real exchange rate and relative consumption.

Choi (2005) presents the first empirical study with positive results for a
structural model of real exchange rate fluctuation. In parallel to this paper, Choi examines the relation between real exchange rates and relative consumptions by augmenting the standard consumption specification with relative trade flows and traded sector consumptions. In the B&S model the real exchange rate is linked to consumption by (1). Most research has focused on testing this directly, implicitly or explicitly assuming complete markets. Choi takes a different approach. If aggregate consumption, $C$, is a CES aggregate of traded, $C_T$, and nontraded, $C_N$, consumption $C = [\alpha_T C_T^\rho + \alpha_N C_N^\rho]^{\frac{1}{\rho}}$ and utility of aggregate consumption is $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, where $\gamma$ is relative risk aversion, then the individual first order optimality conditions are of the form

$$P_T = \lambda \alpha_T C^{1-\gamma-\rho} C_T^{\rho-1}$$

for the home country and

$$P_T = \lambda \alpha_T (C^*)^{1-\gamma-\rho} (C^*)^{\rho-1}$$

for the foreign, with $P_T$ the price of tradeables, $P_N$ the price of non-tradeables, $\alpha$ constant and * representing foreign country variables. B&S assume complete markets so for them $\lambda$ is a constant. Choi (2005) assumes these can vary and uses (2), (3) and (1) to get a new empirical specification for the real exchange rate:

$$RER_t = \left( \frac{\alpha_T}{\alpha_T^*} \right) \left( \frac{C_t}{C_t^*} \right)^{1-\rho} \left( \frac{C_{T,t}}{C_{T,t}^*} \right)^{\rho-1}$$

This specification does well at matching the data for the real exchange rate between the U.S. and Canada, Japan, the U.K. and France. Choi (2005) reports correlations in excess of 0.8 for the real exchange rate and the right hand side of (4). Thus, a model with variable $\lambda$’s would be capable of generating the right empirical relationship. There are two ways to generate variable $\lambda$’s – taste shocks and limited commitment in asset markets. This paper builds a model with the latter.

### 3 The Model

This section will present the general equilibrium two sector, two country model. I consider two environments – complete markets and limited commitment in asset markets. The complete markets economy is a two country version of Backus and Smith (1993) and is used as a benchmark. The limited commitment environment builds on the work in several recent papers introducing this asset market friction in general equilibrium economies to reconcile asset price and consumption data. Trade frictions in the form of nontraded goods are introduced in both environments to account for the large and persistent deviations.

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from purchasing power parity. The asset market friction is introduced in an attempt to make the movements in real exchange rate and relative consumptions empirically sound.

### 3.1 Complete Markets

#### 3.1.1 Setup

The theoretical world is a two-country stochastic exchange economy with each country modeled as a single representative agent. Each agent lives infinitely many periods starting with date 0. At each date \( t \), for \( t = 0, 1, ..., \infty \), an event \( s_t \) occurs that is observed by all agents. I assume that \( s_t \) is drawn from a finite set. The vector of events from time 0 to \( t \) is known as a history and is denoted by \( s_t = (s_0, s_1, ..., s_t) \). The history completely describes the state of the economy at any date \( t \). The probability of any state \( s_t \) is denoted by \( \pi(s_t) \).

In each state there are three goods – one traded good and one nontraded good in each country. Respective foreign country variables will be indexed by *. Denote the endowments of traded and non-traded goods in the home country by \( Y_T(s_t) \) and \( Y_N(s_t) \), respectively. Then the world endowment of traded good is given by \( Y_w = Y_T + Y_T^* \). The corresponding home country consumption quantities are given by \( C_T(s_t) \) and \( C_N(s_t) \). Prices of the goods at time \( t \) are given by \( q_T(s_t) \), for the world’s single traded good, \( q_N(s_t) \), for the home country and \( q_{N*}(s_t) \), for the foreign country nontraded goods. The quantities and prices are functions of the realization of the state of the economy. For simplicity of notation the states will be dropped in the formal presentation of most equations.

The representative consumer in each country has preferences defined by discounted sum of expected utility of aggregate consumption at every date. Formally, lifetime utility is given by

\[
U = \sum_{t=0}^{\infty} \sum_{s_t} \beta^t \pi(s_t) u(c(s_t))
\]

where \( \beta \in (0, 1) \) is the time discount factor and \( u(c) \) is a period utility function. The function \( u(c) \) is identical for both countries and is assumed to be homothetic, quasi-concave, continuous and twice differentiable. The functional form used in this paper is CRRA utility

\[
u(c) = \frac{c^{1-\gamma}}{1-\gamma}
\]

where \( \gamma \) is the coefficient of relative risk aversion. Homotheticity of the utility function is sufficient for existence of aggregate price and quantity indices \( P(q_t^T, q_t^N) \) and \( C(C_t^T, C_t^N) \) such that at utility maximizing quantities the following condition holds

\[
P(q_t^T, q_t^N) C(C_t^T, C_t^N) = q_t^T C_t^T + q_t^N C_t^N
\]
Aggregate consumption for a state $s^t$ is given by a linear, homogeneous function

$$C(s^t) = C(C^T(s^t), C^N(s^t))$$

The functional form is assumed to be a constant elasticity of substitution (CES) aggregate of individual consumptions of traded and nontraded goods:

$$C(C^T, C^N) = \left[ \alpha (C^T)^\rho + (1 - \alpha) (C^N)^\rho \right]^{\frac{1}{\rho}}$$ (7)

The elasticity of substitution between the traded and nontraded consumptions is given by $\frac{1}{\rho}$. With the CES consumption aggregator the corresponding aggregate price index is given by

$$P(q^T, q^N) = \left[ \alpha (\frac{1}{\rho}) (q^T)^{\frac{1}{\rho}} + (1 - \alpha) (\frac{1}{\rho}) (q^N)^{\frac{1}{\rho}} \right]^{\frac{\rho}{\rho-1}}$$ (8)

The asset market in the economy consists of Arrow securities. At every date these Arrow securities promise to deliver one unit of aggregate consumption in a particular realization of the state next period. The price of these securities, given state $s^t$ today, is composed of the price of aggregate consumption today and the intertemporal price of consumption next period in current consumption terms:

$$P(q^T(s^t), q^N(s^t)) Q(s_{t+1}|s^t)$$

The home country holdings of each security is denoted by $w(s^t)$.

### 3.1.2 Equilibrium

To define the equilibrium in this economy it is first necessary to present the problem facing each country.

**Definition 1 (Agent Problem)** Given a sequence of prices

$$\{q^T(s^t), q^N(s^t), \text{ and } P(q^T(s^t), q^N(s^t)) Q(s_{t+1}|s^t)\}_{t=0}^{\infty}$$

the representative consumer in the home country chooses a state-contingent consumption plan $\{C^T(s^t), C^N(s^t)\}$ and asset holdings $w(s^{t+1})$ to maximize the discounted sum of period utility

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c(s^t))$$

subject to a sequence of budget constraints

$$\{q^T(s^t) C^T(s^t) + q^N(s^t) C^N(s^t) + \sum_{s^{t+1}} P(s^t) Q(s_{t+1}|s^t) w(s^{t+1}) \} \leq q^T(s^t) Y^T(s^t) + q^N(s^t) Y^N(s^t) + P(s^t) w(s^t) \}_{t=0}^{\infty}$$ (9)

The foreign country budget constraint is identical but with foreign country variables, indexed by $*$.\(^9\)

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\(^9\)Formally there is also a natural borrowing constraint that rules out Ponzi schemes but it is ignored in the analysis since in practice the Inada condition implies that the borrowing constraint never binds because marginal utility explodes as consumption tends to zero.
Given the definition of the agent problem the competitive equilibrium is defined as follows:

**Definition 2 (Competitive Equilibrium)** The competitive equilibrium in this economy is given by a sequence of prices

\[
\begin{align*}
\left\{ \begin{array}{c}
q^T (s^t), q^N (s^t), P (q^T (s^t), q^N (s^t)) Q (s_{t+1}|s^t), \\
q^{N*} (s^t), P^* (q^T (s^t), q^{N*} (s^t)) Q^* (s_{t+1}|s^t)
\end{array} \right\}_{t=0}^{\infty}
\end{align*}
\]

and an allocation of consumption and bond holdings

\[
\left\{ C^T (s^t), C^N (s^t), w_{t+1}^T (s^t), C^{T*} (s^t), C^{N*} (s^t), w_{t+1}^* (s^t) \right\}_{t=0}^{\infty}
\]

such that

1. They solve the home country agent problem
2. They solve the foreign country agent problem
3. Goods markets clear in all states
   \[
   C^T (s^t) + C^{T*} (s^t) = Y^T (s^t) + Y^{T*} (s^t) = Y^w (s^t)
   \]
4. Arbitrage is ruled out in securities markets
   \[
   Q (s_{t+1}) = RER (s^t) * Q^* (s_{t+1})
   \]

What are the implications of competitive equilibrium for prices and quantities? The exchange rate and interest rates are the aggregate prices of particular interest in this economy and are defined as follows. Given aggregate price indexes \( P, P^* \) the exchange rate in the home country is defined as:

\[
RER_t \equiv \frac{P^* (q^T_t, q^{N*}_t)}{P (q^T_t, q^N_t)}
\]

The real interest rate is defined from the prices of risk-free bonds paying one unit of the aggregate consumption bundle in each state. A risk free bond is constructed as a portfolio of Arrow securities. The price in aggregate consumption terms of each Arrow security is given by

\[
Q (s_{t+1}|s^t) = \beta u_1 (C (s_{t+1}|s^t)) \pi (s_{t+1}|s^t) / u_1 (C (s^t))
\]

where \( u_1 (.) \) is the derivative of the utility function with respect to aggregate consumption. The price of a risk-free bond is then given by the sum of Arrow prices scaled by the price of a unit of aggregate consumption:

\[
d (s^t) = \sum_{s^t+1} P (q^T (s^t), q^N (s^t)) Q (s_{t+1}|s^t)
\]
Finally the real interest rate is given by

\[ R(s^t) = \frac{1}{d(s^t)} - 1 \]  

(15)

At each date bond prices and real interest rates could vary across countries if price indexes did, however in this paper price indexes in the two countries have the same functional form with identical \( \alpha \) and \( \rho \) parameters – a standard simplifying assumption.

In practice the equilibrium quantities are computed using the Negishi-Mantel algorithm as an outcome of a social planning problem. Given planner weights \( \lambda, \lambda^* \) the planner solves

\[
\max_{\{C^T, C^N\}} [\lambda U + \lambda^* U^*] \\
= \max_{\{C^T, C^N\}} \left[ \lambda \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi (s^t) u \left( C \left( C^T (s^t), C^N (s^t) \right) \right) + \lambda^* \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi (s^t) u \left( C^* \left( C^T^* (s^t), C^N^* (s^t) \right) \right) \right]
\]

subject to equation (7) and the resource constraints (10). Let the Lagrange multipliers be given by \( \beta^t \pi (s^t) q^T (s^t) \) for the traded good and \( \beta^t \pi (s^t) q^N (s^t) \) for the non-traded good in the home country. Then for every history \( s^t \) the first-order optimality conditions in the home country are

\[
\lambda u_1 (C) C_1 \left( C^T, C^N \right) = q^T \\
\lambda u_1 (C) C_2 \left( C^T, C^N \right) = q^N
\]

which, using CRRA utility (6) and the CES consumption aggregator (7), turn out to be

\[
\alpha \lambda C^{1-\gamma-\rho} (C^T)^{\rho-1} = q^T \\
\lambda (1-\alpha) C^{1-\gamma-\rho} (C^N)^{\rho-1} = q^N
\]

(16)  

(17)

Combine the aggregate price and consumption indexes, given by equations (8, 7), with the first-order conditions (16, 17) to find

\[ P = \lambda c^{-\gamma} \]

It follows that the real exchange rate (for the home country) is given by

\[ RER_t = \frac{\lambda^*}{\lambda} \left( \frac{C_t}{C^*_t} \right)^\gamma \]  

(18)

Clearly the correlation between exchange rate and relative consumption is unity since the planner weights are fixed.

### 3.2 Limited Commitment

#### 3.2.1 Setup

The only source of difference with the complete markets setup is an asset market where an agent can default on the Arrow securities. The new asset structure is
captured through an additional borrowing constraint for each country at every date. This setup extends Alvarez and Jermann (2001) to a multi-good setting. The agent problem is now defined as follows.

**Definition 3 (Agent Problem)** Given a sequence of prices

\[ \{q^T(s^t), q^N(s^t), P(q^T(s^t), q^N(s^t)) Q(s_{t+1}|s^t)\}_{t=0}^{\infty} \]

the representative consumer in the home country chooses a state-contingent consumption plan \( C^T(s^t), C^N(s^t) \) and asset holdings \( w(s^{t+1}) \) to maximize the discounted sum of period utility

\[ \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c(s^t)) \]

subject to a sequence of budget constraints

\[
\{q^T(s^t)C^T(s^t) + q^N(s^t)C^N(s^t) + \sum_{s^{t+1}} P(s^t) Q(s^t|s^{t+1}) w(s^{t+1}) \leq q^T(s^t)Y^T(s^t) + q^N(s^t)Y^N(s^t) + P(s^t) w(s^t)\}_{t=0}^{\infty}
\]

and borrowing constraints

\[ \{w(s^{t+1}) \geq B(s^{t+1})\}_{t=0}^{\infty} \tag{19} \]

The foreign country budget and borrowing constraints are identical, but with foreign country variables indexed by *.

The borrowing limit on the right hand side of equation (19) is defined in two steps. First, rewrite the agent problem recursively as

\[
V(B(s), s) = \max [u(C(s)) + \beta E[V(B(s'), s'|s)]]
\]

s.t. \( q^T(s)C^T(s) + q^N(s)C^N(s) + \sum_{s'} P(s') Q(s'|s) w(s'|s) \leq q^T(s)Y^T(s) + q^N(s)Y^N(s) + P(s) w(s) \)

and

\[ w(s'|s) \geq B(s'|s) \]

Second, let the value from permanent financial autarky starting in state \( s \) be given by \( V^A(s) \). The borrowing limit under limited commitment in asset markets is defined by

\[ V(B(s'), s') = V^A(s') \]

These borrowing limits ensure that the constraint is binding in states where the complete markets allocation would result in default. Alvarez and Jermann call them borrowing limits that "are not too tight" because they bind just enough to prevent equilibrium default in asset markets. Hence the borrowing constraint
endogenizes the possibility of default and aims at finding an allocation where default will not occur in equilibrium. Clearly the autarky allocation always satisfies these borrowing constraints, but the main interest is in finding another allocation under which trade does take place and enforcement constraints are satisfied in equilibrium.

3.2.2 Equilibrium

The competitive equilibrium in this economy is defined as follows:

**Definition 4 (Competitive Equilibrium)** The competitive equilibrium in this economy is given by a sequence of prices

\[
\{q^T(s^t), q^N(s^t), P(q^T(s^t), q^N(s^t))Q(s_{t+1}|s^t), q^{N*}(s^t), P^*(q^T(s^t), q^{N*}(s^t))Q^*(s_{t+1}|s^t)\}_{t=0}^{\infty}
\]

and an allocation of consumption and bond holdings

\[
\{C^T(s^t), C^N(s^t), w(s^{t+1}), C^{T*}(s^t), C^{N*}(s^t), w^*(s^{t+1})\}_{t=0}^{\infty}
\]

such that

1. They solve the home country agent problem
2. They solve the foreign country agent problem
3. Goods markets clear in all states
   \[
   C^T(s^t) + C^{T*}(s^t) = Y^T(s^t) + Y^{T*}(s^t) = Y^w(s^t)
   
   C^N(s^t) = Y^N(s^t)
   
   C^{N*}(s^t) = Y^{N*}(s^t)
   \]
4. Arbitrage is ruled out in securities markets

\[
Q(s^{t+1}) = RER(s^t) * Q^*(s^{t+1})
\]

What are the implications of competitive equilibrium for prices and quantities in this setting? The home country real exchange rate and real interest rate are defined by (12) and (15), as in complete markets. However, the fundamental behavior of prices and quantities requires a closer characterization of equilibrium.

Because there is a full set of securities traded, equilibrium quantities can be solved using the Negishi-Mantel algorithm, as was done in the complete markets case. The asset market structure is incorporated in the planning problem through a sequence of enforcement constraints. At each date \( t \) enforcement constraints for the home country are of the form:

\[
\sum_{k=t}^{\infty} \sum_{s^k} \beta^{k-t} u(C(s^k)) \pi(s^k|s^t) \geq V^A(s^t)
\]

(20)
where $\pi(s^k|s^t)$ is the conditional probability of $s^k$ given $s^t$, with $\pi(s^t|s^t) = 1$, and $V^A(s^t)$ is the value of autarky from $s^t$ onward. This value of autarky is given by

$$V^A_i(s^t) = \sum_{k=t}^{\infty} \sum_{s} \beta^{k-t} \pi(s^k|s^t) u^i(c_i(y_i(s_k), x_i(s_k)))$$  \hspace{1cm} (21)$$

The value of sticking with the trade arrangement is given by

$$\sum_{k=t}^{\infty} \sum_{s^k} \beta^{k-t} u(C(s^k)) \pi(s^k|s^t)$$

This is a useful simplifying technique since it allows the direct computation of quantities without tracking prices. Prices can be backed out using the equilibrium quantities and appropriate relations. The planning problem setup is as follows. Attach planner weights $\lambda (\lambda^*)$ to the home (foreign) country and solve

$$\max_{\{CT, CN\}} \left[ \lambda \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(C(s^t)) + \lambda^* \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(C^*(s^t)) \right]$$ \hspace{1cm} (22)$$

subject to enforcement (20) and resource constraints (10).

A major complication with solving the competitive equilibrium allocation under limited commitment is that future choice variables - traded and non-traded goods consumption - enter the current enforcement constraints. This problem with future choice variables in current constraints makes inapplicable the standard solution method based on dynamic programming. To deal with this problem Marcet and Marimon (1997) show that extending the state space by introducing co-state variables makes the problem tractable. Kehoe and Perri (2002) show that in an economy similar to the described in this paper the correct co-state variable is the ratio of the sums of Lagrange multipliers on the enforcement constraints. The way to incorporate the co-state variable and recast the problem is developed as follows. Let $\beta^t \pi(s^t) \theta(s^t)$ denote the Lagrange multiplier on the enforcement constraint (20) in the home country and $\beta^t \pi(s^t) \theta(s^t)$ the corresponding multiplier in the foreign country. The Lagrangian is the sum of (22), the standard resource constraints (10) with shadow prices as multipliers and each country’s enforcement constraint term

$$\beta^t \pi(s^t) \theta(s^t) \left[ \sum_{k=t}^{\infty} \sum_{s^k} \beta^{k-t} u(C(s^k)) \pi(s^k|s^t) - V^A(s^t) \right]$$

The Lagrangian can be regrouped as the standard terms related to the resource constraints and

$$\sum_{t=0}^{\infty} \sum_{s^t} \left( \mu(s^t) \beta^t \pi(s^t) \left[ u(C(s^t)) - V^A(s^t) \right] + \mu^*(s^t) \beta^t \pi(s^t) \left[ u(C^*(s^t)) - V^{A*}(s^t) \right] \right)$$
where $\mu(s^t)$ is the co-state variable defined by its law of motion as

$$\mu(s^t) = \mu(s^{t-1}) + \theta(s^t)$$  \hspace{1cm} (23)

with initial condition $\mu(s^{-1}) = \lambda$. The $\mu$'s are simply the sum of the initial planner weight and the multipliers on the past enforcement constraints along the history $s^t$.

Given the co-state variable the implications of equilibrium for prices and quantities can be assessed. The first order optimality conditions in every history $s^t$ are

$$\mu(s^t) u_1(C(s^t)) C_1(C^T(s^t), C^N(s^t)) = q^T(s^t)$$  \hspace{1cm} (24)

$$\mu(s^t) u_1(C(s^t)) C_2(C^T(s^t), C^N(s^t)) = q^N(s^t)$$  \hspace{1cm} (25)

Using the functional forms for utility and aggregate price and consumption indexes (6, 8, 7) with the first order conditions, the aggregate price in the home country is given by

$$P(s^t) = \mu(s^t) [C(s^t)]^{-\gamma}$$

In the foreign country the relation holds with * indexing the foreign country variables. Therefore the real exchange rate, defined by equation (12), is

$$RER(s^t) = \frac{\mu^*(s^t)}{\mu(s^t)} \left( \frac{C(s^t)}{C^*(s^t)} \right)^{\gamma}$$  \hspace{1cm} (26)

The co-state variables, $\mu$, are time varying and depend on the binding pattern and history of the enforcement constraints. Thus while in complete markets the real exchange rate was given by (18) where the weights assigned to each agent were fixed, in the limited commitment case the weights are allowed to vary. This extra term will reduce the correlation between fluctuations in relative consumption and real exchange rate if the country co-state variable is positively correlated with that country’s aggregate consumption. The real interest rate is defined by (14, 15). In contrast to complete markets the price, in real terms, of each Arrow security is given by

$$Q(s_{t+1}|s^t) = \max_i \left[ \frac{u_1^i(C^i(s_{t+1}|s^t)) \pi(s_{t+1}|s^t)}{u_1^i(C^N(s^t))} \right]$$  \hspace{1cm} (27)

where $i$ denotes the home or foreign country and $u_1(.)$ is the derivative of the utility function with respect to aggregate consumption. Note that the Arrow price is given by the highest marginal rate of intertemporal substitution among the two countries. This is a feature of models with limited commitment in asset markets where marginal rates of intertemporal substitution are no longer equalized for all agents. This happens because the borrowing constraints are binding for one of the agents. One implication is a time-varying risk premium – a feature of the data that is empirically documented but not explored in this model$^{10}$.

4 Mechanics

In this section I lay out the mechanics and the intuition for the theoretical link between real exchange rate and relative consumption in the models with limited commitment and complete markets. To examine the inner workings of the models I run the following two-part experiment. In the first part of the experiment I restrict fluctuations to the endowment of traded goods only, leaving the supply of nontraded goods fixed. In the second part of the experiment I fix the supply of traded goods and let the supply of nontraded goods vary. This experiment lets me evaluate the mechanics of the relationship between real exchange rate and relative consumption implied by the two models. I examine real exchange rate and relative consumption fluctuations by analyzing supply and demand for traded and nontraded goods. I illustrate the analysis by simulating impulse responses, presented in figures 2-5, for the relevant variables in the two model environments.  

To carry out the experiment I impose the following structure on the models. The model is fully described by the equations presented in section 3 and the parameter values for the time discount factor ($\beta$), risk aversion ($\gamma$), share of traded good in aggregate consumption ($\alpha$), elasticity of substitution between traded and nontraded goods ($\frac{1}{1+\frac{1}{\rho}}$) and the endowment process. Parameter values, while chosen arbitrarily, implicitly take some features of the data into account. The endowment process is assumed to be a three state Markov chain. The Markov chain is given by the transition probability matrix, $P$, and the vector of endowments, $y$. The transition probability matrix is obtained by 

$$P = \psi I + (1 - \psi) \Pi$$

where $I$ is the identity matrix, $\Pi$ is a 3x3 matrix with all entries equal to 1/3 and $\psi$ is the persistence. I examine $\psi = \{0.7, 0.99\}$ but will limit the presentation to $\psi = 0.7$ since temporary, rather than permanent, shocks are the main interest of this experiment. The endowments are summarized by a 3x4 matrix where each column represents the endowment of traded or nontraded good in the home or foreign country. When the supply of traded good is allowed to fluctuate the endowment in the home country is given by $Y = [y + \delta, y, y]'$ and in the foreign country by $Y^* = [y, y, y + \delta]'$. Thus in state 1 the home country receives a high shock to the supply of tradeables, in state 3 it is the foreign country that gets it and in state 2 the endowments in the two countries are the equalized. The supply of nontraded good is given by a vector of ones in each country. In the second part of the experiment the situation is reversed. The supply of traded good in both countries is given by a vector of ones while the supply of nontraded goods is given by $Y$ and $Y^*$ in the home and foreign countries, respectively. The values are set at $y = 1$ and $\delta = \sqrt{0.3}$. The other parameter values are set at $\beta = 0.65$, $\gamma = 2$, $\alpha = 0.5$, $\rho \in \{-1.27, 0, 0.8\}$.  

Shocks to traded good endowment lead to real exchange rate appreciation under limited commitment but have no effect under complete markets. 

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\[11\] The impulse response is calculated by simulating the economy for 100 periods and averaging over 1,500 simulations. This is standard practice, e.g. Tesar (1995)
the traded sector a change in the supply of the traded good is a pure supply shock, with no change in the demand schedule. Thus it results in a decrease of the price of traded good and an increase in world traded good consumption. Because the traded and nontraded goods are imperfect substitutes a supply shock in the traded sector translates into a demand shock in the nontraded sector. Therefore, since the supply of nontraded goods is fixed a positive shock to the world supply of the traded good that pushes up the demand schedule for nontraded goods results in a higher price of nontraded good. The price increase in the nontraded sector in a country is proportional to the magnitude of the traded sector supply shock in that country after trade. Under complete markets the extra supply of traded good is split evenly between the two countries thus leading to equal increases in the price of nontraded goods in the two countries. The real exchange rate is proportional to the ratio of nontraded good prices since the only variation in the aggregate price index in each country comes from the price of nontraded goods and the real exchange rate is the ratio of the aggregate price indexes. Thus in the case of complete markets the ratio of nontraded good prices is unchanged due to perfect sharing of the supply shock in the traded sector. However under limited commitment the price of nontraded goods in the foreign country increases less than the domestic one. When the home country has a higher supply of the traded good it has an incentive to default on its payments to the foreign country. In equilibrium this results in a binding borrowing constraint for the foreign country and thus an uneven sharing of the extra supply of traded good. Since the nontraded sector demand shock is proportional to the magnitude of the traded good consumption increase in that country, the price of nontradeables increases more in the home country than in the foreign one. Thus the real exchange rate level drops – a real appreciation for the home country.

Shocks to the nontraded good endowment lead to real exchange rate depreciation in both models. Nontraded sector shocks are basically idiosyncratic, affecting only the home price of nontraded goods. Thus an increase in the home country supply of the nontraded good results in a lower price of nontraded goods at home but not abroad causing the ratio of foreign to domestic nontraded goods prices to increase. One way in which nontraded sector supply shocks are not exactly idiosyncratic is that they are also traded sector demand shocks because of the imperfect substitutability of traded and nontraded goods consumption. However this affects trade but not the exchange rate since the price of tradeables is the same in both countries. The real exchange rate increases in proportion to the increased ratio of nontraded good prices, a real depreciation for the home country. The mechanics are similar for both models because nontraded sector shocks are uninsurable.

Supply shocks to either traded or nontraded good result in higher relative consumptions for the model with limited commitment while only shocks to the nontraded sector have that effect in the complete markets setting. In the model with complete markets a shock to the home country supply of traded good is fully risk-shared and results in equal and higher traded goods consumptions at home and abroad. However with limited commitment the same shock is not
fully risk-shared and the extra traded good consumption is distributed unevenly, with more going to the home country. This results in higher aggregate consumption since nontraded good consumption is unchanged in both countries. Thus the consumption ratio, which is the ratio of domestic to foreign aggregate consumptions, increases under limited commitment. Clearly, with complete markets this ratio remains unchanged. Shocks to the home country supply of nontraded good result in higher consumption of nontraded good and lower consumption of traded good, which is traded abroad. The decrease in the traded good consumption does not perfectly offset the increase in nontraded good consumption, resulting in higher domestic aggregate consumption relative to its foreign counterpart. Thus relative consumption increases. Once again the models behave similarly for nontraded goods because these shocks are cannot be directly insured in financial markets.

The effect of supply shocks in the traded and nontraded sectors on the correlation of real exchange rate and relative consumption is different for the two models. The models behave similarly for high (low) shock to the supply of nontraded goods, with both real exchange rate and relative consumption increasing (decreasing). Since shocks to the supply of traded goods have no effect with complete markets this is the mechanics behind the B&S perfect correlation result. However in the limited commitment model a high (low) shock to the supply of traded good results in higher (lower) relative consumption and lower (higher) real exchange rate. Thus when there are shocks to the supply of both traded and nontraded goods their effects cancel out leading to the close to zero correlation between relative consumption and real exchange rate in the model with limited commitment.

5 Calibration

5.1 Setup

Here I use parameters consistent with an "average" industrialized country to asses model performance by comparing it with the empirical evidence discussed in section 2. In order to have a quantitative exercise several features of the data, such as production, government, growth, money and nominal rigidities, would need to be incorporated. However the qualitative behavior of the economy is captured by the model presented here. Thus, rather than a way to get quantitative predictions the calibration is meant as a disciplining device testing whether realistic parameter values generate the desired results in the model with limited commitment.

Table 4 summarizes the parameter values. I use annual data for 21 countries described in section 2. The data on consumption and GDP are broken down into traded and notraded sectors following Stockman and Tesar (1995). The natural logarithm of the sectoral GDP and consumption data are detrended using a linear filter with a country-specific trend to make them stationary.\[12\]

\[12\]For the results of a calibration based on HP filtered data see the appendix.
The real exchange rate level is detrended using the linear filter to maintain consistency within the dataset. The model is fully described by the set of equations presented in section 3 and the parameter values for the time discount factor ($\beta$), risk aversion ($\gamma$), share of traded good in aggregate consumption ($\alpha$), elasticity of substitution between traded and nontraded goods ($\frac{1}{1-p}$) and the endowment process.

The model parameters are set as follows. Consider first the preference parameters. Risk aversion determines how the variation in income affects consumption and trade and the degree of real exchange rate volatility relative to consumption. Common numbers for risk aversion range between 1 and 6. In this paper risk aversion is set at $\gamma = 2$ or $\gamma = 4$, with the latter producing more realistic volatilities of real exchange rate relative to consumption. The time discount factor determines the interest rate and is set at $\beta = 0.95$ to generate an interest rate between 4 and 5%. This is the number commonly used in the literature. Next consider the parameters that relate to the consumption aggregator. The share of traded goods is taken from Stockman and Tesar (1995) and set at $\alpha = 0.5$. The elasticity of substitution is also taken from Stockman and Tesar and set at $\epsilon = 0.44$ implying $\frac{1}{1-p} = 2.7$. The endowment process is calibrated in a two step procedure. First, I estimate a Vector Auto Regression (VAR) on a vector of detrended sectoral GDP in both countries. I impose symmetry restrictions and examine two cases – spillover and no spillover. Formally the estimation equation for the VAR is given by

$$
\begin{bmatrix}
Y^T_{t}
Y^N_{t}
Y^T_{t-1}
Y^N_{t-1}
\end{bmatrix}
= 
\begin{bmatrix}
b_1
b_2
\end{bmatrix}
+ 
\begin{bmatrix}
a_1 & a_2 & a_3 & 0 
 a_4 & a_5 & a_6 & 0 
 a_3 & 0 & a_1 & a_2 
 a_6 & 0 & a_4 & a_5 
\end{bmatrix}
\begin{bmatrix}
Y^T_{t-1}
Y^N_{t-1}
Y^T_{t-2}
Y^N_{t-2}
\end{bmatrix}
+ 
\begin{bmatrix}
\epsilon^T_{t}
\epsilon^N_{t}
\epsilon^T_{t-1}
\epsilon^N_{t-1}
\end{bmatrix}
$$

(28)

where the innovations are normally distributed with variance-covariance $\Sigma$. The no spillover case is a coefficient restriction of $a_3 = a_6 = 0$. Second, I follow Tauchen (1986) in approximating the VAR by a discrete Markov chain using maximum likelihood. Thus I have a total of four calibrations (two values of risk aversion x two types of spillover) to consider. The results of the VAR’s are presented in the appendix. The VAR makes apparent two features of the filtered GDP data. They are characterized by high persistence, particularly in the nontraded sector combined with relatively low variation. This is consistent with balanced growth among the developed economies.

5.2 Results

Tables 5 and 6 compare the quantitative implications of the complete markets and limited commitment models. The models can be compared with the data by referring to tables 1-3. The correlation of relative consumption and real exchange rate is -0.37 in the limited commitment economy compared to a mean of 0.049 in the data and 1 in complete markets. While the model predicts a lower
correlation of relative consumption and real exchange rate than the data, these results are encouraging when compared to results in the literature. In particular, Chari, Kehoe and McGrattan (2002) find unitary correlations for both complete markets and asset markets restricted to uncontingent bonds. Stockman and Tesar (1995) declare success when using taste shocks that generate the correlation of 0.32.

How does the model perform along other dimensions? The correlation of consumptions across countries is lower under limited commitment than under complete markets, but still higher than the correlation of GDP in contrast to the data. The empirical correlation of consumption across countries has a mean of 0.176 while the cross country correlation of gdp is 0.837. In the model with complete markets the consumption correlation is 0.926 while in the limited commitment model it is 0.561 when the cross country correlation of gdp is 0.555. These findings are in line with Kehoe and Perri (2002) who show that the model with limited commitment is more successful at matching moments of international business cycle data than the complete markets or a model with uncontingent bonds. On the dimension of deviations from PPP the limited commitment model performance is similar to the complete markets model because it is driven by the nontraded good sector where consumption is given by the endowment. The models match the data well with the half-life of real exchange rate innovations at roughly 19 months for both models compared to 27 months in the data.

6 Conclusion

The main contribution of this paper is to show that building a structural model of real exchange rate determination should rely on the interaction of trade and asset market frictions. Introducing limited commitment in asset markets breaks the tight link between real exchange rate and relative consumption. In a calibrated economy the correlation of real exchange rate and relative consumption is in line with the empirical evidence. This is an improvement on the previous attempts to model the relationship that rely on trade frictions or nominal rigidities and monetary shocks.

There are several extensions of the model that are appealing. An empirical examination that applies and tests the model would need to determine how to incorporate asset market frictions into an exchange rate model and whether or not default in asset markets is in fact the right friction. One possible way to check whether default is the right friction is to back out default probabilities from Credit Default Swap (CDS) data and check whether it influences the relationship between relative consumption and real exchange rate. A theoretical extension would introduce production into the model in order to closely scrutinize its performance against empirical regularities and test whether such a model is capable of delivering sound quantitative predictions.
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7 Appendix

7.1 Data Appendix

The dataset consists of annual data for twenty one OECD countries covering the years 1980-2000. The countries in the sample are Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Iceland, Italy, Japan, Korea, the Netherlands, New Zealand, Norway, Portugal, Spain, Sweden, the United Kingdom and the United States. Household consumption, exports of goods and services, and imports of goods and services, both nominal and real, come from OECD Annual National Accounts, Level I. Traded consumption is the sum of Food and Non-Alcoholic Beverage consumption, Alcoholic Beverage, Tobacco and Narcotics consumption, Clothing and Footwear consumption, Communications consumption, and Recreation and Culture consumptions from the OECD Annual National Accounts, Level II. Nontraded consumption is the sum of Housing, Water, Electricity, Gas and Other Fuels consumption, Furnishings, Household Equipment and Routine Maintenance of the House consumption, Health consumption, Transport consumption, Education consumption, Restaurant and Hotel consumption, and Miscellaneous Goods and Services consumption from the OECD Annual National Accounts, Level II. Total Value Added (gdp) is taken from OECD STAN database. Traded value added (gdp) is the sum of Agriculture, Hunting, Forestry and Fishing value added, Mining and Quarrying value added, Total Manufacturing value added, Wholesale and Retail Trade value added, and Transportation, Storage and Communication value added from OECD STAN database. Nontraded value added (gdp) is the sum of Electricity, Gas and Water Supply value added, Construction value added, Finance, Insurance, Real Estate and Business Services value added, and Community, Social and Personal Services value added from the OECD STAN database. Real variables are constructed by deflating total and the disaggregated value added series using appropriate deflators from the OECD STAN database. Exchange Rate (AVG) is taken from the World Bank WDI database. The PPP deflator for the exchange rate is taken from the World Bank WDI database. International dollar exchange rates are the product of the Exchange Rate and the PPP deflator.
7.2 Tables and Figures

Figure 1: Levels of \( C/C^* \) and RER for US-Japan
Fig. 2: Impulse Response, Limited Commitment, Nontraded Shocks
Fig. 3: Impulse Response, Complete Markets, Nontraded Shocks
Fig. 4: Impulse Response, Limited Commitment, Traded Shocks
Fig. 5: Impulse Response, Complete Markets, Traded Shocks
Table 1: Correlation of Real Exchange Rate and Consumption Ratio (source: OECD Annual National Accounts)

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<td>0.000 (0.004)</td>
</tr>
<tr>
<td>France</td>
<td>0.309 (0.180)</td>
<td>-0.003 (0.004)</td>
</tr>
<tr>
<td>Germany</td>
<td>0.367 (0.200)  *</td>
<td>-0.004 (0.005)</td>
</tr>
<tr>
<td>Greece</td>
<td>0.370 (0.203)  *</td>
<td>-0.007 (0.005)</td>
</tr>
<tr>
<td>Iceland</td>
<td>23.795 (12.557) *</td>
<td>-0.188 (0.275)</td>
</tr>
<tr>
<td>Italy</td>
<td>0.322 (0.215)</td>
<td>-0.004 (0.006)</td>
</tr>
<tr>
<td>Japan</td>
<td>45.134 (30.760)</td>
<td>-1.025 (0.923)</td>
</tr>
<tr>
<td>Korea</td>
<td>489.511 (237.339) **</td>
<td>-6.220 (5.388)</td>
</tr>
<tr>
<td>Netherlands</td>
<td>0.439 (0.199)  **</td>
<td>-0.005 (0.005)</td>
</tr>
<tr>
<td>New Zealand</td>
<td>0.402 (0.388)</td>
<td>0.000 (0.009)</td>
</tr>
<tr>
<td>Norway</td>
<td>1.936 (1.460)</td>
<td>-0.005 (0.029)</td>
</tr>
<tr>
<td>Portugal</td>
<td>0.391 (0.233)</td>
<td>-0.010 (0.007)</td>
</tr>
<tr>
<td>Spain</td>
<td>0.309 (0.190)</td>
<td>-0.005 (0.006)</td>
</tr>
<tr>
<td>Sweden</td>
<td>2.438 (1.467)</td>
<td>-0.023 (0.038)</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>0.314 (0.112)  **</td>
<td>-0.005 (0.002)</td>
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<tr>
<td>Country</td>
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<td>CT</td>
</tr>
<tr>
<td>-------------</td>
<td>-----</td>
<td>-----</td>
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<tr>
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<td>0.268</td>
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</tr>
<tr>
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<tr>
<td>Belgium</td>
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<tr>
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<tr>
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</tr>
<tr>
<td>France</td>
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</tr>
<tr>
<td>Germany</td>
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<tr>
<td>Greece</td>
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<tr>
<td>Iceland</td>
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<td>n.a.</td>
</tr>
<tr>
<td>Italy</td>
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<td>-0.118</td>
</tr>
<tr>
<td>Japan</td>
<td>-0.222</td>
<td>-0.706</td>
</tr>
<tr>
<td>Korea</td>
<td>-0.317</td>
<td>-0.699</td>
</tr>
<tr>
<td>Netherlands</td>
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<tr>
<td>New Zealand</td>
<td>0.489</td>
<td>n.a.</td>
</tr>
<tr>
<td>Norway</td>
<td>0.522</td>
<td>0.791</td>
</tr>
<tr>
<td>Portugal</td>
<td>-0.414</td>
<td>n.a.</td>
</tr>
<tr>
<td>Spain</td>
<td>0.036</td>
<td>n.a.</td>
</tr>
<tr>
<td>Sweden</td>
<td>0.678</td>
<td>n.a.</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>0.720</td>
<td>0.878</td>
</tr>
<tr>
<td>Mean</td>
<td>0.176</td>
<td>0.128</td>
</tr>
</tbody>
</table>

Table 4: Calibration Parameterization

<table>
<thead>
<tr>
<th>Calibration</th>
<th>γ</th>
<th>α</th>
<th>ρ</th>
<th>β</th>
<th>spillover</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
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<td>-1.27</td>
<td>0.95</td>
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<tr>
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<td>2</td>
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</tr>
<tr>
<td>3</td>
<td>4</td>
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<td>-1.27</td>
<td>0.95</td>
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</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.5</td>
<td>-1.27</td>
<td>0.95</td>
<td>no</td>
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</table>
### Table 5: Calibration Results, Cross Country Correlations (Linear Filtered Data)

<table>
<thead>
<tr>
<th>Calibration</th>
<th>Limited Commitment</th>
<th>Complete Markets</th>
<th>Endowment Process</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$RER$</td>
<td>$C, C^*$</td>
<td>$C^T, C^{T*}$</td>
</tr>
<tr>
<td>1</td>
<td>-0.763</td>
<td>0.596</td>
<td>0.510</td>
</tr>
<tr>
<td>2</td>
<td>-0.378</td>
<td>0.559</td>
<td>0.419</td>
</tr>
<tr>
<td>3</td>
<td>-0.764</td>
<td>0.594</td>
<td>0.507</td>
</tr>
<tr>
<td>4</td>
<td>-0.374</td>
<td>0.561</td>
<td>0.428</td>
</tr>
</tbody>
</table>

### Table 6: Halflife of Real Exchange Rate Innovations

<table>
<thead>
<tr>
<th>Calibration</th>
<th>Limited Commitment</th>
<th>Complete Markets</th>
<th>Endowment Process</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_0$</td>
<td>$\beta_1$</td>
<td>$\beta_2 RER_{t-1}$</td>
</tr>
<tr>
<td>1</td>
<td>0.355 (0.003)***</td>
<td>-0.000 (0.000)</td>
<td>0.645 (0.003)***</td>
</tr>
<tr>
<td>2</td>
<td>0.320 (0.003)***</td>
<td>-0.000 (0.000)</td>
<td>0.679 (0.003)***</td>
</tr>
<tr>
<td>3</td>
<td>0.356 (0.003)***</td>
<td>-0.000 (0.000)</td>
<td>0.644 (0.003)***</td>
</tr>
<tr>
<td>4</td>
<td>0.326 (0.003)***</td>
<td>0.000 (0.000)</td>
<td>0.674 (0.003)***</td>
</tr>
</tbody>
</table>
7.3 Calibration Based on Hodrick Prescott Filtered Data

The Hodrick Prescott (HP) filter is the standard tool for isolating the cyclical component of macroeconomic variables in the business cycle literature. However Baxter and King (1995) argue that there are several drawbacks to using the HP filter. One issue is that the filtered series tend to have significantly diminished persistence. In the model with limited commitment in asset markets, discussed in this paper, lower persistence in the income process leads to higher gains from risk sharing, resulting in equilibrium allocations closer to complete markets (full risk sharing). This exercise can be thought of as a robustness check. In general the results are robust to this specification, in fact they are improved.

The overall methodology is unchanged from the calibration based on the linear filter discussed in the paper. The HP filtered GDP by sector is used to calibrate the endowment process as described in section (5). The results of the calibrated model are then HP filtered and compared with the HP filtered data. The numbers are as follows:

<table>
<thead>
<tr>
<th>Country</th>
<th>Level</th>
<th>HP Filter (λ=100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>0.231</td>
<td>0.091</td>
</tr>
<tr>
<td>Austria</td>
<td>0.142</td>
<td>-0.108</td>
</tr>
<tr>
<td>Belgium</td>
<td>-0.084</td>
<td>0.170</td>
</tr>
<tr>
<td>Canada</td>
<td>-0.061</td>
<td>0.036</td>
</tr>
<tr>
<td>Denmark</td>
<td>-0.457</td>
<td>0.310</td>
</tr>
<tr>
<td>Finland</td>
<td>-0.081</td>
<td>0.331</td>
</tr>
<tr>
<td>France</td>
<td>-0.135</td>
<td>0.315</td>
</tr>
<tr>
<td>Germany</td>
<td>0.310</td>
<td>0.221</td>
</tr>
<tr>
<td>Greece</td>
<td>-0.565</td>
<td>-0.187</td>
</tr>
<tr>
<td>Iceland</td>
<td>-0.221</td>
<td>0.398</td>
</tr>
<tr>
<td>Italy</td>
<td>-0.446</td>
<td>0.531</td>
</tr>
<tr>
<td>Japan</td>
<td>0.763</td>
<td>-0.206</td>
</tr>
<tr>
<td>Korea</td>
<td>0.591</td>
<td>0.732</td>
</tr>
<tr>
<td>Netherlands</td>
<td>0.545</td>
<td>0.387</td>
</tr>
<tr>
<td>New Zealand</td>
<td>-0.449</td>
<td>-0.374</td>
</tr>
<tr>
<td>Norway</td>
<td>-0.126</td>
<td>0.206</td>
</tr>
<tr>
<td>Portugal</td>
<td>-0.635</td>
<td>0.189</td>
</tr>
<tr>
<td>Spain</td>
<td>-0.367</td>
<td>0.476</td>
</tr>
<tr>
<td>Sweden</td>
<td>-0.294</td>
<td>0.200</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>-0.368</td>
<td>0.382</td>
</tr>
<tr>
<td>Mean</td>
<td>-0.085</td>
<td>0.205</td>
</tr>
</tbody>
</table>
Table 7: Correlation with US Variable
(HP filtered data (λ=100), source: OECD ANA and STAN databases)

<table>
<thead>
<tr>
<th>Country</th>
<th>C</th>
<th>CT</th>
<th>CN</th>
<th>Y</th>
<th>YT</th>
<th>YN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>0.392</td>
<td>n.a.</td>
<td>n.a.</td>
<td>0.397</td>
<td>0.248</td>
<td>0.310</td>
</tr>
<tr>
<td>Austria</td>
<td>-0.549</td>
<td>-0.139</td>
<td>-0.517</td>
<td>0.753</td>
<td>0.869</td>
<td>0.571</td>
</tr>
<tr>
<td>Belgium</td>
<td>-0.115</td>
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<td>n.a.</td>
<td>0.615</td>
<td>0.186</td>
<td>0.805</td>
</tr>
<tr>
<td>Canada</td>
<td>0.819</td>
<td>0.937</td>
<td>0.833</td>
<td>0.835</td>
<td>0.381</td>
<td>0.646</td>
</tr>
<tr>
<td>Denmark</td>
<td>0.345</td>
<td>0.254</td>
<td>0.178</td>
<td>0.899</td>
<td>0.598</td>
<td>0.682</td>
</tr>
<tr>
<td>Finland</td>
<td>0.652</td>
<td>0.677</td>
<td>0.646</td>
<td>0.718</td>
<td>0.485</td>
<td>0.554</td>
</tr>
<tr>
<td>France</td>
<td>0.310</td>
<td>0.306</td>
<td>0.384</td>
<td>0.742</td>
<td>0.343</td>
<td>0.982</td>
</tr>
<tr>
<td>Germany</td>
<td>-0.220</td>
<td>n.a.</td>
<td>n.a.</td>
<td>0.392</td>
<td>0.538</td>
<td>0.133</td>
</tr>
<tr>
<td>Greece</td>
<td>-0.293</td>
<td>-0.0412</td>
<td>-0.124</td>
<td>0.641</td>
<td>0.736</td>
<td>0.363</td>
</tr>
<tr>
<td>Iceland</td>
<td>0.527</td>
<td>n.a.</td>
<td>n.a.</td>
<td>0.569</td>
<td>0.325</td>
<td>0.782</td>
</tr>
<tr>
<td>Italy</td>
<td>0.295</td>
<td>0.435</td>
<td>0.268</td>
<td>0.835</td>
<td>0.469</td>
<td>0.900</td>
</tr>
<tr>
<td>Japan</td>
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<td>-0.442</td>
<td>-0.348</td>
<td>0.645</td>
<td>0.685</td>
<td>0.506</td>
</tr>
<tr>
<td>Korea</td>
<td>-0.377</td>
<td>-0.323</td>
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<td>0.609</td>
<td>0.779</td>
<td>-0.012</td>
</tr>
<tr>
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<td>n.a.</td>
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<tr>
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<td>n.a.</td>
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<td>-0.000</td>
<td>0.211</td>
</tr>
<tr>
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<td>0.612</td>
<td>0.295</td>
<td>0.467</td>
<td>0.392</td>
<td>0.458</td>
</tr>
<tr>
<td>Portugal</td>
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<td>n.a.</td>
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<td>0.224</td>
<td>0.595</td>
</tr>
<tr>
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<td>n.a.</td>
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<td>0.434</td>
<td>0.779</td>
</tr>
<tr>
<td>Sweden</td>
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<td>n.a.</td>
<td>n.a.</td>
<td>0.575</td>
<td>0.562</td>
<td>0.209</td>
</tr>
<tr>
<td>United Kingdom</td>
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<td>0.797</td>
<td>0.718</td>
<td>0.747</td>
<td>0.868</td>
<td>0.200</td>
</tr>
<tr>
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<td>0.184</td>
<td>0.624</td>
<td>0.488</td>
<td>0.505</td>
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</table>

Table 8: Calibration Parameterizations

<table>
<thead>
<tr>
<th>Clabiration</th>
<th>γ</th>
<th>α</th>
<th>ρ</th>
<th>β</th>
<th>spillover</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
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</tr>
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<td>0.95</td>
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</tr>
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<td>8</td>
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</table>
Table 9: Calibration Results, Cross Country Correlations (HP filtered data)

<table>
<thead>
<tr>
<th>Calibration</th>
<th>Limited Commitment</th>
<th>Complete Markets</th>
<th>Endowment Process</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RER, $\frac{L}{L}$</td>
<td>$C, C^<em>$, $C^T, C^{T</em>}$, $C^N, C^{N*}$</td>
<td>$Y, Y^<em>, Y^T, Y^{T</em>}, Y^N, Y^{N*}$</td>
</tr>
<tr>
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<td>0.779 0.695 0.562</td>
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</tr>
<tr>
<td>6</td>
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<td>0.785 0.702 0.521</td>
<td>1 0.843 1 0.521</td>
</tr>
<tr>
<td>7</td>
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<td>0.784 0.670 0.560</td>
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<td>1 0.926 1 0.526</td>
</tr>
</tbody>
</table>
7.4 Computational Appendix

The computation procedure is performed as follows. Let the state space be defined by \( v = \{s, z, \mu_1, \mu_2\} \) where \( s \) is the natural state and \( z \) the co-state. The natural state, \( s = \{x_1, y_1, x_2, y_2\} \), consists of the individual endowments of traded \((y_i)\) and non-traded \((x_i)\) goods. The co-state variable \((z)\), is a function of the Lagrange multipliers on the enforcement constraints recursively defined by its law of motion and initial conditions \( \mu_2 = \lambda_2, \mu_1 = \lambda_1 \) given by planner weights. The law of motion for the co-state into the current state is given by

\[
z'(v) = \frac{\mu_2 (v)}{\mu_1 (v)} = \frac{\mu_2 + \theta_2}{\mu_1 + \theta_1}
\]

where \( \mu_i \) is the individual co-state last period and \( \theta_i \) is the Lagrange multiplier on the enforcement constraint in the current period. The timing is thus somewhat unorthodox in that \( z'(v) \) denotes the co-state in the current period, however last period co-state and current period enforcement constraint are necessary to update the co-state variable to the current period. The algorithm computes optimal policy functions for current consumptions of traded and non-traded goods \( a_i(v), b_i(v) \), the current relative co-state \( z'(v) \) and the individual current co-states \( z'_i(v) \). For computational purposes define value functions

\[
W_i(v) = u_i(c_i(a_i(v), b_i(v))) + \beta \sum_{s' \in S} \pi(s'|s) W_i(s')
\]

All these functions must satisfy the resource constraints (10), enforcement constraints (20), optimality conditions (24,25) and the law of motion for the co-state variables (23). To implement the procedure a grid \( V \) is defined over the state-space and the solution is restricted to the class of functions that take arbitrary values for every \( v \in V \) and are equal to piecewise bilinear interpolation for values \( v \notin V \). Thus the solutions are exact at the grid points \( v \in V \) and are approximations for values off the grid. The size of the interval between each grid point and the shape of the true function being approximated govern the accuracy of the approximation, with smaller intervals and less curvature improving accuracy.

The solution is obtained using a policy function iteration algorithm modified to handle enforcement constraints. Start off solving the planning problem (22) by an initial guess on the policy functions given by the solution to the problem without enforcement constraints. Denote these initial policy functions by \( a^0_i(v), b^0_i(v), z^0_i(v), W^0_i(v) \) for every \( v \in V \). Given the initial guess, the resource constraints, enforcement constraints and the optimality condition a new set of policy functions is found, denoted by \( a^1_i(v), b^1_i(v), z^1_i(v), W^1_i(v) \) for every \( v \in V \). The new set of functions is found as follows. Since the binding pattern of the enforcement constraints is \textit{a priori} unknown three possibilities are considered: none are binding, home country constraint is binding or the foreign country constraint is binding. (The two constraints cannot be binding for both countries simultaneously given the endowment process in this model).
For each $v$ compute the allocations assuming that the enforcement constraints aren’t binding this period and check if they satisfy the enforcement constraints

$$u_i(c_i(a_i(v), b_i(v))) + \beta \sum_{s'} \pi(s'|s) W^0_i(v') \geq V^A_i(v)$$  \hspace{1cm} (30)

for each country. If the allocations do satisfy both constraints (30) then set $\theta_i = 0$ for both countries yielding a new co-state from (29) and define new value functions by the left side of (30). If the allocation does not satisfy the enforcement constraint of country 1, then set $\theta_2 = 0$ and write the enforcement constraint of country 1 with equality

$$u_1(c_1(a_1(v), b_1(v))) + \beta \sum_{s'} \pi(s'|s) W^1_1(v') = V^A_1(v)$$  \hspace{1cm} (31)

Define the new value function by the left side of (30) for country 2 and the left side of (31) for country 1. The process for computing the value function if the enforcement constraint of country 2 is violated is identical to the above, but the countries are switched. The procedure is repeated for every $v \in V$ and the difference between the vectors $\{a^0_i(v), b^0_i(v), z^0_i(v), W^0_i(v)\}$ and $\{a^0_i(v), b^0_i(v), z^0_i(v), W^0_i(v)\}$ is compared to a small positive number, $\psi$. If the difference is smaller than $\psi$, then computation is complete. However, if the difference is higher than $\psi$, then redefine the initial guess by the newly obtained value and policy functions and repeat the process. The solution is obtained by iterating until the vectors converge.