LIABILITIES

EFFECTS

Fall 2002
Surplus = Assets - Liabilities

\[ S_1 = A_1 - L_1 \quad \text{at one} \]

\[ S_0 = A_0 - L_0 \quad \text{at zero} \]

\[ \frac{S_1}{S_0} = (1 + r_s) = \frac{A_1 - L_1}{A_0 - L_0} \]

\[ = \frac{1}{A_0} \begin{pmatrix} A_0 \\ S_0 \end{pmatrix} - \frac{1}{L_0} \begin{pmatrix} L_0 \\ S_0 \end{pmatrix} \]

\[ = (1 + r) \frac{A}{S_0} - (1 + r) \frac{L}{S_0} \]
\[ R_S = \left( \begin{array}{c} A \\ 0 \\ S \\ 0 \end{array} \right) r_A - \left( \begin{array}{c} L \\ 0 \\ S \\ 0 \end{array} \right) r_L \]

\[ \text{Mean } \bar{R}_S \]

\[ \bar{R}_S = \left( \begin{array}{c} A \\ 0 \\ S \\ 0 \end{array} \right) \bar{R}_A - \left( \begin{array}{c} L \\ 0 \\ S \\ 0 \end{array} \right) \bar{R}_L \]

\[ \text{Variance } \sigma^2_S \]

\[ \sigma^2_S = \left( \begin{array}{c} A \\ 0 \\ S \\ 0 \end{array} \right)^2 \sigma^2_A + \left( \begin{array}{c} L \\ 0 \\ S \\ 0 \end{array} \right)^2 \sigma^2_L - 2 \left( \begin{array}{c} A \\ 0 \\ S \\ 0 \end{array} \right) \left( \begin{array}{c} L \\ 0 \\ S \\ 0 \end{array} \right) \rho_{AL} \sigma_A \sigma_L \]
What can manager influence:

1. Can not affect \( A_0, L_0 \) or \( S_0 \)
2. Can not affect \( \bar{r}_L \) or \( \sigma_L \)
3. Can only affect \( \rho_{AL}, \bar{R}_A \) and \( \sigma_A \)

Note if \( \rho_{AL} = 0 \) can concentrate on asset allocation and ignore liabilities but being conscience that mean return and variance of surplus are affected by liabilities when looking at trade off.

What if \( \rho_{AL} \neq 0 \)?

1. What are desirable assets?

Those assets that serve hedge functions.
Unlike normal mean variance, portfolio manager needs to be concerned with $\sigma_A, \overline{R}_A$, and $\rho_{AL}$.

Can view as three-dimensional with the following properties:

1. $\max \overline{R}_A$

2. $\min \sigma_A$

3. $\max \rho_{AL}$

Efficient set defined over these three.
Consider, however, trade off. Terms that influence variance under managers control are:

\[
\sigma_s^2 = \left( \frac{A}{S} \right)^2 \sigma_A^2 + \left( \frac{L}{S} \right)^2 \sigma_L^2 - 2 \frac{A}{S} \frac{L}{S} \rho_{AL} \sigma_A \sigma_L
\]

\[
\sigma_s^2 = \left( \frac{L}{S} \right)^2 \sigma_L^2 + \frac{2A}{2S} \sigma_L^2 \left[ \left( \frac{1}{2\sigma_L} \right) \sigma_A^2 - \rho_{AL} \sigma_A \right]
\]

Note first term and term in front of brackets is not under manager's control, thus manager can only control

\[
\left[ \left( \frac{1}{2\sigma_L} \right) \sigma_A^2 - \rho_{AL} \sigma_A \right]
\]

as long as this term is constant risk unchanged and this term measures trade-off.
For example, assume:

1. \( A_0 = 100 \)
2. \( L_0 = 80 \)
3. \( \sigma_L = 20 \)

then \[
\frac{1}{2\sigma} \frac{A_0}{L_0} = \frac{1}{40} \frac{100}{80} = \frac{1}{32}
\]

and I would be equally happy to choose a portfolio with \( \sigma_A^2 \) up 10 if \( \rho_{AL} \sigma_A \) was also up \( \left( \frac{1}{32} \right) \cdot 10 \).

This explicit tradeoff allows me to collapse the choice into mean return and standard deviation.
Example 2:

Shape and TINIC suggests giving less than full credit to the risk reducing aspects of liabilities because, among other reasons, uncertainty in their estimated value.

<table>
<thead>
<tr>
<th>Asset Classes</th>
<th>$\rho_{iL}$</th>
<th>$\sigma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intermediate Bonds</td>
<td>.20</td>
<td>8</td>
</tr>
<tr>
<td>Growth Stocks</td>
<td>.65</td>
<td>20</td>
</tr>
</tbody>
</table>

$$A_0 = 100$$

$$L_0 = 80$$

$$\sigma_L = 20$$
\[ \frac{1}{2} A_0 \sigma_A^2 \]

Intermediate

\[ \frac{110064}{28020} = 2 \]

Growth

\[ \frac{1100400}{28020} = 12.5 \]

and term in parenthesis

\[(2 - 1.6) = .4\]

\[(12.5 - 13) = -.5\]

and the higher "risk" asset is actually less risky. It is possible, however, that both assets should enter.
Note last term in brackets is:

\[
\frac{\text{cov}\left(R_A R_L\right)}{\sigma_A \sigma_L} \sigma_A
\]

or

\[
\frac{\text{cov}(R_A R_L)}{\sigma_L}
\]

But

\[
\text{Cov}\left(R_A R_L\right) = \Sigma_i \text{Cov}(R_i R_L)
\]

so individual assets enter linearly. The affect on last term in brackets can be looked at one term at a time. However, this does not hold for \( \sigma_A \).