1. Assumptions that residuals are uncorrelated from a single-index model turns out to be inaccurate. This assumption can be improved by assuming additional indexes.

2. Once again multi-index models have developed a life of their own beyond their original purpose of estimating covariances.
Consider two index model:

\[ r_{it} = a_i + b_{il} I_{it} + b_{12} I_{2t} + e_{it} \]

Where:

1. \( E(I_{lt}, I_{2t}) = 0 \)
2. \( E(e_{it}) = 0 \)
3. \( E(e_{it} I_{lt} \text{ or } I_{2t}) = 0 \)
4. \( E(e_{it} e_{jt}) = 0 \)
5. \( a_i, b_{il} \text{ and } b_{i2} \) are constant.

Only real assumption is \( E(e_{it} e_{jt}) = 0 \)
Mean return:

\[ E\left(r_{it}\right) = a_i + b_{i1}\bar{I}_1 + b_{i2}\bar{I}_2 \]

\[ \text{Var}\left(r_{it}\right) = b_{i1}^2\sigma^2_{\bar{I}_1} + b_{i2}^2\sigma^2_{\bar{I}_2} + \sigma^2_{\epsilon_i} \]

in other words:

\[ \sigma^2_{\epsilon_i} = b_{i2}^2\sigma^2_{\bar{I}_2} + \sigma^2_{\epsilon_i} \]

\[ \text{cov}\left(r_{i} r_{j}\right) = b_{i1} b_{j1}\sigma^2_{\bar{I}_1} + b_{i2} b_{j2}\sigma^2_{\bar{I}_2} \]
Note:

1. if add $E\left( e_i e_j \right)$ get historical.

2. assuming $E\left( \varepsilon_i \varepsilon_j \right) = b_{i2} b_{j2} \sigma^2 I2 + 0$
Generalizing to more than two indexes is straightforward.

**Types of indexes:**

(1). Statistically derived

(2). Portfolios of securities

   A. S&P, H-L, B-M
   B. S&P, H-L, B-M, Bonds

   Fama French
   E&G

(3). Economic Factors

   A. ΔIP, surprise in inflation
   B. Surprise GNP

(4). Market and Industry

   A. S&P
   B. Industry factors
In bond area:

\[ R_{it} = \bar{R}_i + OAS - D_i \frac{\Delta r_i}{1 + r_i} - V_i dV_i + C_i \text{(spread)} + e_i \]
Issues:

(1). Real influences

(2). Parsimonious
Uses:

(1). Covariance structure

(2). Selection of exposure

(3). Return attribution

(4). Portfolio evaluation