THE MULTIPLICITY OF RATES

September 1999
The Multiplicity of Rates

\[ r_{t_0't_1't_2} \]

define

\[ t_0 = \text{time of commitment} \]

\[ t_2 = \text{time money repayed} \]

\[ t_1 = \text{time money lent} \]

Definition:

Spot rate is the rate of interest on a bond with one payment and where commitment and lending date are the same.

Forward rate is the rate of interest on a loan where the commitment date and date money is lent are different.
Relationship of Forwards and Spots

Equivalent:

\[(1 + \frac{r_{001}}{2})(1 + \frac{r_{012}}{2}) = (1 + \frac{r_{002}}{2})^2\]

\[1 + \frac{r_{012}}{2} = \frac{(1 + \frac{r_{002}}{2})^2}{(1 + \frac{r_{001}}{2})}\]

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

\[\leftarrow \frac{r_{001}}{2} \rightarrow \leftarrow \frac{r_{012}}{2} \rightarrow\]

\[\leftarrow \leftarrow \frac{r_{002}}{2} \rightarrow \rightarrow\]
\[(1 + \frac{r_{002}}{2})^3 = (1 + \frac{r_{002}}{2})^2 (1 + \frac{r_{023}}{2}) \]

\[1 + \frac{r_{023}}{2} = \frac{(1 + \frac{r_{003}}{2})^3}{(1 + \frac{r_{002}}{2})^2}\]
Pure discount bond is a bond with single repayment date a zero.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-90</td>
<td>+100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-82</td>
<td>+100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-76</td>
<td>+100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-100</td>
<td></td>
<td>+100</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+140</td>
</tr>
</tbody>
</table>

discount function \( D(t) = \frac{1}{(1 + \frac{r_{00t}}{2})^t} \)
What Determines Spot Rates or Term Structure Theories?

We talk about term structure theories to explain this curve:

a. Pure expectations
b. Liquidity
c. Preferred habitat
d. Market segmentation

The importance of these from an investment point of view is: do we earn an excess return on average by holding long bonds?

(1) Expectations theory = No term premium
(2) Liquidity = Positive term Premium
(3) Preferred habitat = ?
(4) Market Segmentation = ?
How does this come about?

1. Expectations Theory

We know

\[
\left( 1 + \frac{r_{002}}{2} \right)^2 = \left( \frac{1 + r_{001}}{2} \right) \left( \frac{1 + r_{012}}{2} \right)
\]

 expectations theory \[ \frac{r_{112}}{2} = \frac{r_{012}}{2} = \frac{(1 + \frac{r_{002}}{2})^2}{(1 + \frac{r_{001}}{2})} \]
(1) one period return

a. One period bond earns

\[
\frac{r_{001}}{2}
\]

b. two period sold after one period

\[
\frac{r_{112}}{2}
\]

At 1, if expectations realized then must offer and sell for

\[
\frac{(1 + \frac{r_{002}}{2})^2}{(1 + \frac{r_{112}}{2})} = \frac{(1 + \frac{r_{002}}{2})^2}{(1 + \frac{r_{012}}{2})} = (1 + \frac{r_{001}}{2})
\]

\[
\text{first period return } \frac{r_{001}}{2}
\]
Law of one Price: Two identical items can't sell at different prices.

<table>
<thead>
<tr>
<th>Time</th>
<th>Bond A</th>
<th>Bond B</th>
<th>Bond C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>110</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>115</td>
</tr>
</tbody>
</table>

\[(100)x_A + 10\ x_B = 15\]

\[110\ x_B = 115\]

\[X_B = \frac{23}{22} \quad X_A = \frac{1}{22}\]
check

\[ 100 \left( \frac{1}{22} \right) + 10 \left( \frac{23}{22} \right) = 15 \]

\[ 110 \left( \frac{23}{22} \right) = 115 \]

Thus

\[
\begin{bmatrix}
1 \text{ Bond A} \\
23 \text{ Bond B}
\end{bmatrix}
\text{(is equivalent to)}
\begin{bmatrix}
22 \\
\text{Bond C}
\end{bmatrix}
\]

implication

\[ P_C = \frac{1}{22} P_A + \frac{23}{22} P_B \]
Note:

A coupon paying bond can always be viewed as a portfolio of pure discount instruments.

To see this, it is convenient to assume all pure discount instruments mature at $1.

Consider

<table>
<thead>
<tr>
<th>Time</th>
<th>Price of pure discount instruments</th>
<th>Maturity value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$P_1$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$P_2$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$P_3$</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$P_4$</td>
<td>1</td>
</tr>
</tbody>
</table>
Consider Bond with cash flows as follows:

<table>
<thead>
<tr>
<th>Time</th>
<th>cf</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>110</td>
</tr>
</tbody>
</table>

Then this bond can be viewed as equivalent to 10 of 1, 10 of 2, 10 of 3, and 110 of 4.

Price is \(10P_1 + 10P_2 + 10P_3 + 110P_4\)

Since \(P_t = \frac{1}{(1 + \frac{r00t}{2})^t}\)

\[
\text{Price} = \frac{10}{(1 + \frac{r001}{2})} + \frac{10}{(1 + \frac{r002}{2})^2} + \frac{10}{(1 + \frac{r003}{2})^3} + \frac{110}{(1 + \frac{r004}{2})^4}
\]
Implication:

Bonds must be priced as if each cash flow is discounted at spot rate (if law of one price holds).

Thus spot rates are the basic building blocks of bond valuation.
Two ways to spot mispriced bonds:

1. Directly looking for swaps.

2. Indirectly by computing spot rates and comparing "equilibrium" price to traded price.
**Swap Example:**

The prior example was an example of a swap.

A second example:

<table>
<thead>
<tr>
<th>Time</th>
<th>Bond A</th>
<th>Bond B</th>
<th>Bond C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>105</td>
<td>105</td>
<td>110</td>
</tr>
<tr>
<td>Price</td>
<td>92</td>
<td>96</td>
<td>103</td>
</tr>
</tbody>
</table>

Have three bonds and two time periods; thus can always create one bond from other two. Which is chosen is arbitrary. Assume matching cash flows of c.
\[ x_A = \text{amount in } A \]

\[ x_B = \text{amount in } B \]

To match cash flows:

\[ 100 \ x_A + 5x_B = 10 \]

\[ 105 \ x_B = \frac{110}{105} \]

\[ x_A = \frac{10 - 5 \left( \frac{110}{105} \right)}{105} \]
Check ...

\[ 100 \left( \frac{5}{105} \right) + 5 \left( \frac{110}{105} \right) = 10 \]

\[ 105 \left( \frac{110}{105} \right) = 110 \]

Combination Costs ...

\[ 92 \left( \frac{5}{105} \right) + 96 \left( \frac{110}{105} \right) = 104.9524 \]

Since C costs 103, have an arbitrage. Holders of A and B should sell and buy C.
Estimating Spot Rates

Note:

In what follows I will be using price to stand for invoice price or what one pays for the bond. Invoice price is quoted price plus accrued interest.

\[
P = \frac{cf(1)}{(1 + \frac{r_{001}}{2})} + \frac{cf(2)}{(1 + \frac{r_{002}}{2})^2} + \frac{cf(3)}{(1 + \frac{r_{003}}{2})^3}
\]

\[
D(t) = \frac{1}{(1 + \frac{r_{00t}}{2})^t}
\]

\[
P = CF(1)D(1) + CF(2)D(2) + CF(3)D(3)
\]

Problem is to estimate D(t)
Using zeros

• observe two period zero at $920

\[ 920 = \frac{1000}{(1 + \frac{r_{002}}{2})^2} \]

\[ \frac{r_{002}}{2} = \left( \frac{1000}{920} \right)^2 - 1 \]

\[ r_{002} = 8.514 \]
Using sequence of bonds

<table>
<thead>
<tr>
<th>Bond</th>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-930</td>
<td>1000</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>-800</td>
<td>100</td>
<td>1100</td>
</tr>
<tr>
<td>C</td>
<td>-700</td>
<td>80</td>
<td>80</td>
</tr>
</tbody>
</table>

*use bond A to determine $r_{001}$ or $D(1)$*

+$930 = 1000 \ D(1)$
\[ D(1) = \frac{930}{1000} \]

*Use bond B and D(1) to determine \( r_{002} \) or D(2)*

\[ 800 = 100D(1) + 1100D(2) \]

\[ 800 = 93 + 1100D(2) \]

\[ 800 = 100 \left( \frac{930}{1000} \right) + 1100D(2) \]

\[ D(2) = \frac{707}{1100} \]
Use bond C and D(1) and D(2) to obtain D(3)

$$700 = 80D(1) + 80D(2) + 1080D(3)$$

$$700 = 80\left(\frac{930}{1000}\right) + 80\left(\frac{707}{1100}\right) + 1080D(3)$$

$$700 = 74.4 + 51.42 + 1080D(3)$$

$$700 = 125.82 + 1080D(3)$$

$$D(3) = \frac{574.18}{1080}$$
Problem is sensitive to bonds chosen

Law of one price may not hold perfectly because

(1) Non synchronous trading

(2) Bid-ask spread

(3) Bonds out of equilibrium
Solution: Use average estimate

\[ P_i = cf(1)D(1) + cf(2)D(2) + cf(3)D(3) + e_i \]

Estimate regression

Important often to constrain so conforms to what we KNOW must be true.

\[ D(t) > D(t + 1) \]

Problem with this procedure is that it depends on bonds having same coupon dates. Alternative is to estimate a function.
assume smooth and concave from above
Example:

\[ D(t) = C_0 + C_1 t + C_2 t^2 \]

\[ P_i = \sum_t c f(t) D(t) + e_i \]

\[ P_i = \sum_t c f(t)(c_0 + c_1 t + C_2 t^2) + e_i \]

\[ P_i = C_0 [\sum c f(t)] + c_1 [\sum c f(t) t] + c_2 [\sum c f(t) t^2] + e_i \]

estimate \( c_0, c_1, c_2 \)
Putting in Tax Term

**historically**

Adjusted cash flows to after tax

\[ p_i = cf(1)D(1) + cf(2)D(2) + cf(3)D(3) + (100 - p_0)t + e_i \]

**currently**

For individuals mainly postponement thus differential taxation function of maturity as well as price relative to Par

**clientele effects**

?
Review

I. Terms

a. Spot rate
b. Forward rate
c. Pure discount bond or zero
d. Law of one price

II. Concepts

a. That default free bonds should be priced by calculating present value of cash flows at the spot rate
b. Bond swaps
c. Bond arbitrage
d. Consequences of term structure theories

III. Calculations

a. Spot rates
b. Forward rates
c. Bond swaps
d. Checking for arbitrage
Problems

I. Given the following invoice prices and cash flows, what are the spot rates and forward rates?

<table>
<thead>
<tr>
<th>Period</th>
<th>Invoice Price</th>
<th>Cash flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>960</td>
<td>1000</td>
</tr>
<tr>
<td>1</td>
<td>920</td>
<td>1000</td>
</tr>
<tr>
<td>2</td>
<td>880</td>
<td>1000</td>
</tr>
</tbody>
</table>

Answer:

\[
960 = \frac{1000}{(1 + \frac{r_{001}}{2})}
\]

\[
\frac{1000}{960} = 1 + \frac{r_{001}}{2}
\]

\[
r_{001} = 8.33
\]
\[
920 = \frac{1000}{(1 + \frac{r_{002}}{2})^2}
\]

\[
\frac{r_{002}}{2} = \left(\frac{1000}{920}\right)^2 - 1
\]

\[
r_{002} = 8.51
\]

\[
880 = \frac{1000}{(1 + \frac{r_{003}}{2})^3}
\]
\[
\frac{r_{003}}{2} = \left( \frac{1000}{880} \right)^3 - 1
\]

\[r_{003} = 8.71\]

\[
1 + \frac{r_{012}}{2} = \left( \frac{1000}{920} \right) = \frac{960}{920}
\]

\[r_{012} = 8.70\]
\[ 1 + \frac{r_{023}}{2} = \frac{\left( \frac{1000}{880} \right)}{\left( \frac{1000}{920} \right)} = \frac{920}{880} \]

\[ r_{023} = 9.09 \]
2. Consider the following:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Invoice Price</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>100</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>102</td>
<td>6</td>
</tr>
<tr>
<td>C</td>
<td>95</td>
<td>102</td>
</tr>
</tbody>
</table>

Is there an arbitrage? What is the swap, if any?

Answer:

Use B and C to match the cash flows of A.

\[ x_C \cdot 102 + x_B \cdot 6 = 5 \]
\[ x_B \cdot 106 = 105 \]

\[ x_B = \frac{105}{106} \]

\[ 102 \cdot x_C = 5 - 6 \left( \frac{105}{106} \right) \]
\[ X_C = \frac{-100}{(106)(102)} \]

Cost of replicating portfolio:

\[ 102 \left( \frac{105}{106} \right) - 95 \left( \frac{100}{106}(102) \right) = 100.1591 \]

Thus A is cheaper. Swap is to sell B and buy A and C in proportions shown. Note \( x_C \) is negative.
Accrued Interest Calculations

Conventions

Different markets have different conventions for calculating accrued interest. The method used in each market is denoted by specifying the method of calculating two values:

\[ d \] is the number of days from the previous coupon payment date to settlement date (or from issue date to settlement date, if the next coupon payment if the first); that is, the number of days over which interest has accrued.

\[ A_y \] is the assumed number of days in one year.

The particular convention used can be Actual/Actual, Actual/365, Actual/360, or 30E/360.

In the name of the convention, the first part of the name denotes the method of computing \( d \); and the second part of the name denotes the method of calculating \( A_y \).
Computing the Accrued Interest

Once $d$ and $A_y$ have been calculated, accrued interest is computed by the formula:

$$I_A = C \frac{d}{A_y}$$

$I_A$ is the accrued interest
$C$ is the annual coupon payment

Calculating $d$

The three methods of calculating $d$ are:

1. "Actual" - Calculate the actual number of days from the previous coupon payment date to the settlement date.
Examples:

There are 10 days from 1/3/93 to 1/13/93
There are 41 days from 1/3/93 to 2/13/93
There are 31 days from 1/1/93 to 2/1/93
There are 28 days from 2/1/93 to 3/1/93
There are 29 days from 2/1/92 to 3/1/92

2. "30" - Calculate the number of days from the previous coupon payment date to the settlement date by assuming 30-day months, as follows:

Let the two dates by M1/D1/Y1 and M2/D2/Y2.
If D1 is 31, change it to 30.
If D2 is 31, change it to 30.
Then \( d = 360(Y_2-Y_1) + 30(M_2-M_1) + (D_2-D_1) \).

(Note: According to this formula, February will always appear to be a 30-day month).

Examples:

There are 10 days from 1/3/93 to 1/13/93
There are 30 days from 1/1/93 to 2/1/93
There are 30 days from 2/1/93 to 3/1/93
There are 30 days from 2/1/92 to 3/1/92
There are 4 days from 2/27/93 to 3/1/93
There are 0 days from 5/30/93 to 5/31/93
There are 2 days from 5/29/93 to 5/31/93

This "30" method is used in the U.S. agencies, corporates, and municipals markets.
3. "30E" - Calculate the number of days from the previous coupon payment date to the settlement date by assuming 30-day months, as follows:

Let the two dates by $M_1/D_1/Y_1$ and $M_2/D_2/Y_2$. If $D_1$ is 31, change it to 30. If $D_2$ is 31, change it to 30. Then

$$d = 360(Y_2-Y_1) + 30(M_2-M_1) + (D_2-D_1).$$

(Note: According to this formula, February will always appear to be a 30-day month.)

Examples:

There are 30 days from 2/1/93 to 3/1/93
There are 30 days from 2/1/92 to 3/1/92
There are 4 days from 2/27/93 to 3/1/93
There are 0 days from 5/30/93 to 5/31/93
There is 1 day from 5/29/93 to 5/31/93

This method, slightly different from the "30" method above, is used in the Eurobond market and many European government and corporate markets.
Calculating $A_y$

The three methods of calculating $A_y$ are:

1. "365" - $A_y$ is equal to 365.

2. "360" - $A_y$ is equal to 360.

3. "Actual" - $A_y$ is equal to the number of days in the current coupon period times the number of coupon payments per year. For a semi-annual coupon, the number of days in the coupon period can range from 181 to 184, so $A_y$ can range from 362 to 368.
Examples:

(1) A 6.75% bond paying semi-annually is traded to settle on July 2, 1993. The previous coupon paid on March 15, 1993, and the next coupon pays on September 15, 1993. If the bond accrues according to the Actual/Actual convention, what is the accrued interest at settlement?

There are 184 actual days in the current coupon period, and there are 109 actual days from the last coupon payment to settlement. Therefore, the accrued interest is:

\[
\frac{109}{2 \times 194} \times 6.75 = 1.999321 \text{ per } 100 \text{ of } \text{face value}
\]

(2) Assume that the bond in example (1) accrues according to the 30/360 convention, instead of Actual/Actual. What would be the accrued interest?

There are 107 "30" days from the last coupon payment to settlement. Therefore, the accrued interest is:

\[
\frac{107}{360} \times 6.75 = 2.006250 \text{ per } 100 \text{ of } \text{face value}
\]
# Accrued Interest - Market Conventions

<table>
<thead>
<tr>
<th>Instrument Type</th>
<th>Accrual Convention</th>
<th>Coupons per Year</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Domestic:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>US Govt. Treasury Bonds</td>
<td>Actual/Actual</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>US Govt. Agency Bonds</td>
<td>30/360</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Corporate Bonds</td>
<td>30/360</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Municipal Bonds</td>
<td>30/360</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>GNMA/FNMA/FLHMC</td>
<td>Actual/360</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td><strong>International:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eurobonds</td>
<td>30E/360</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>German Govt. and Corp. Bonds</td>
<td>30E/360</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Italian Govt. and Corp. Bonds</td>
<td>30E/360</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Japanese Govt. and Corp. Bonds</td>
<td>Actual/365</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Swiss Govt. and Corp. Bonds</td>
<td>30/360</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>UK Govt. and Corp. Bonds</td>
<td>Actual/365</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

1. In the Japanese markets, February 29 is never counted.