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Simple Criteria for Optimal Portfolio Selection with Upper Bounds

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We present a new method for selecting optimal portfolios when upper-bound constraints on investments in individual stocks are present and when the variance-covariance matrix of returns possesses a special structure such as that implied by the standard single-index model. The method differs substantially from the usual nonlinear programming methods used in this context and allows the development of criteria that indicate important characteristics of a stock.

Elsewhere [5, 6] we have shown that simple ranking procedures can be used to produce optimum portfolios if one assumes a single-index model or makes special assumptions about the correlation structure. The ranking procedures are simple enough that the optimum portfolio often can be determined by hand calculations.

In this paper we extend the previous analysis to incorporate upper limits on the fraction of the portfolio that can be held in any stock. Institutions are often restricted by law (and individuals frequently by choice) from placing more than a certain percentage of their funds in any one security. For example, the Investment Company Act of 1940 states that no more than 5% of the funds of an investment company can be invested in any one security. Furthermore, the importance of considering the upper-bound constraints can be seen from an empirical study done by Cohen and Pogue [1]. They report that when selecting portfolios from 75 and 150 stock populations, 66–82% of the securities selected were at their legally defined upper bounds of 5%. Naturally, if management were operating under tighter self-imposed upper bounds (as most managers do), the number of securities at the upper bound would tend to be even larger. Thus the problem of incorporating upper bounds is a serious one.

Imposing upper bounds substantially changes the procedures for solving the portfolio problem by ranking techniques and requires a completely new set of proofs of optimality. Nevertheless, by using particular forms of the variance-covariance matrix, one can derive decision rules for optimal portfolio selection with upper bounds that are much more direct and more easily understood than the usual nonlinear programming methods. Further-
more, one can develop criteria for ranking stocks that make clear the characteristics of a stock that are important to the portfolio manager.

Throughout this paper we assume the existence of a risk-free rate at which lending and borrowing can take place. This is not strictly necessary. In [5] we have shown in general how to move from a solution involving a riskless asset to a solution tracing out the full efficient frontier. We have chosen to assume a risk-free rate because the solution is less complex and it is therefore easier to understand the impact of upper-bound constraints.

We develop simple procedures for two cases: when the single-index model is appropriate and when one can assume a constant-correlation coefficient between all pairs of securities. While the single-index model has received a great deal of attention in the economic literature, recent empirical work indicates that the constant-correlation case is also worth studying. In particular, Elton and Gruber [2] have shown that for several samples:

1. Estimates of future correlation coefficients obtained from the constant-correlation model produced smaller forecast errors than did those obtained from the single-index model. Forecast errors were measured using several 5-year and 1-year periods subsequent to the periods over which the models were fit.

2. Portfolios selected using inputs from the constant-correlation coefficient model offered higher return for the same risk than did portfolios selected on the basis of the single-index model.

Thus, portfolio selection procedures using the constant-correlation model are worth exploring along with those of the single-index model. In fact, from a mathematical point of view, the constant-correlation model can be regarded as a special type of single-index model. To see this, one needs a change in notation (see the appendix). Despite this fact, we explore the implications of the constant-correlation coefficient model first because its algebraic treatment is notionally less involved and thus easier to follow. We do, however, state all necessary formulas for the single-index model in the main text as well, while leaving the algebraic detail to the reader. Throughout we assume that short sales are not allowed. In addition, we make the blanket assumption that the upper-bound constraints permit the investor to find a portfolio whose expected return is strictly greater than the return on the riskless asset. After stating the portfolio selection problem in Section 1, we treat, in Section 2, the portfolio model involving a constant-correlation coefficient, derive the decision rules that produce an optimal portfolio, and illustrate the technique by a numerical example. In Section 2 we treat the single-index model due to Sharpe [10]. Finally, some of the more technical points are proven in the appendix.

1. OPTIMAL PORTFOLIO SELECTION WITH UPPER BOUNDS

If there is a riskless lending and borrowing rate, then the separation theorem [7] holds and the optimum portfolio is the one that maximizes
\[
\theta = \frac{\sum_{i=1}^{n} X_i (R_i - R_f)}{(\sum_{i=1}^{n} \sigma_i^2)^{1/2}}
\]

subject to suitable constraints, where \( \bar{R}_p \) = the expected return on a portfolio of securities; \( R_f \) = the riskless lending and borrowing rate; \( \sigma_p \) = the standard deviation on a portfolio of securities; \( X_i \) = the amount invested in security \( i \); \( \sigma_i \) = the variance of security \( i \); \( \bar{R}_i \) = the expected return on security \( i \); \( n \) = the number of securities considered for inclusion in the portfolio.

If short selling is not allowed, we must introduce the constraints that all \( X_i \geq 0 \). If upper-bound constraints exist, then we have that all \( X_i \leq C_i \), where \( C_i \) are positive constants satisfying \( C_i < 1 \) for \( i = 1, \ldots, n \). Finally, to ensure full investment, we have \( \sum_{i=1}^{n} X_i = 1 \). The optimization problem thus becomes

\[
\max \theta
\]

\[
(P') \quad \sum_{i=1}^{n} X_i = 1
\]

\[
X_i \leq C_i, \quad -X_i \leq 0, \quad i = 1, \ldots, n.
\]

We can, of course, assume that \( \sum_{i=1}^{n} C_i > 1 \).

The problem of optimal portfolio selection when upper bounds are present has been treated by Sharpe [10, 11]. He presents a linear programming approximation to the usual quadratic programming problem that arises in using his single-index model. Our method is not an approximation; it determines the true optimum under the assumptions of the single-index model as well as under the alternative assumption of constant-correlation coefficients.

To bring problem \((P')\) into a more tractable form, we replace it by

\[
\max \theta
\]

\[
(P) \quad X_i \leq C_i, \sum_{i=1}^{n} X_i = 1, \quad -X_i \leq 0, \quad i = 1, \ldots, n.
\]

Every solution to \((P')\) is a solution to \((P)\) and vice versa. For every solution to \((P)\) that is not identical to zero in all components we find a solution to \((P')\) simply by dividing all \( X_i \) by \( \sum_{i=1}^{n} X_i \). Furthermore, as the objective function \( \theta \) is homogeneous of degree zero, it follows readily that the equality \( \sum_{i=1}^{n} X_i = 1 \) can be deleted and that an optimal solution to \((P)\) provides an optimal solution to \((P')\) and vice versa. Observing that the denominator of \( \theta \) is defined with respect to a (positive-definite) variance-covariance matrix, we see that \( \theta \) is pseudoconcave over the domain of positivity of \( \theta \) and, furthermore, that \((P)\) can be solved by solving the associated Kuhn-Tucker conditions. More precisely, the Kuhn-Tucker conditions are both necessary and sufficient for a solution to \((P)\) to be optimal. (See [6] for a detailed argument to this effect.)
2. PORTFOLIO SELECTION WITH A CONSTANT-CORRELATION COEFFICIENT

If we assume that the covariance \( \sigma_{ij} \) between security \( i \) and security \( j \) can be written as \( \sigma_{ij} = \rho \sigma_i \sigma_j \), where the correlation coefficient \( \rho \) is a constant, the optimization problem (P) assumes a special form that can be solved by a simple procedure closely resembling the ranking devices developed in [5] and [6]. Elton and Gruber [2] show empirically that the assumption of constant-correlation coefficients in portfolio selection yields an excellent substitute for the general model. Since negative correlation coefficients are rarely observed, we can assume without loss of generality that the constant \( \rho \) is positive.

If we define \( Z_i = \left( \bar{R}_i - R_F \right)/\sigma_i^2 \), where \( \bar{R}_i \) is the expected return and \( \sigma_i^2 \) is the variance of the optimal portfolio, then the Kuhn-Tucker conditions for the maximization problem (P) with \( \sigma_{ij} = \rho \sigma_i \sigma_j \) for all \( i \neq j \) are as follows:

\[
\begin{align*}
Z_i & = \left[ \bar{R}_i - R_F - \rho \sigma_i, \sum_{j=1}^{n} \sigma_j Z_j - \sigma_i (1 - C_i) \delta_i + \sigma_i \mu_i \right] / \sigma_i^2 (1 - \rho) \\
Z_i & \leq C_i, \sum_{j=1}^{n} Z_j \\
Z_i \mu_i & = 0, \ (Z_i - C_i \sum_{j=1}^{n} Z_j) \delta_i = 0 \\
Z_i & \geq 0, \ \mu_i \geq 0, \ \delta_i \geq 0
\end{align*}
\]

where \( i = 1, \ldots, n \) in (1')–(4'). The problem is thus to find a set of numbers \( Z_i, \mu_i, \delta_i \) satisfying (1')–(4'). Similar to the transformation from \( X_i \) to \( Z_i \) that involves a positivity-preserving factor, we set \( \delta_i = \sigma_i (1 - C_i) \delta_i / \sigma_i^2 (1 - \rho) \) and \( \mu_i = \sigma_i \mu_i / \sigma_i^2 (1 - \rho) \) and replace (1')–(4') by the following equivalent system of equations:

\[
\begin{align*}
Z_i & = \left( \bar{R}_i - R_F \right)/\sigma_i^2 (1 - \rho) - \left[ \rho / (1 - \rho) \sigma_i \right] \sum_{j=1}^{n} \sigma_j Z_j - \delta_i + \mu_i \\
Z_i & \leq C_i, \sum_{j=1}^{n} Z_j \\
Z_i \mu_i & = 0, \ (Z_i - C_i \sum_{j=1}^{n} Z_j) \delta_i = 0 \\
Z_i & \geq 0, \ \mu_i \geq 0, \ \delta_i \geq 0
\end{align*}
\]

Here we have used \( C_i < 1 \) for \( i = 1, \ldots, n \), in order to establish a one-to-one correspondence between the solutions to (1')–(4') and (1)–(4). In practice, this assumption can be weakened to \( C_i \leq 1 \) for some \( i \).

Let \( Z_i, j = 1, \ldots, n \), denote a feasible solution to (1)–(4) and set \( \phi = \rho \sum_{j=1}^{n} \sigma_j Z_j \). We obtain from equation (1)

\[
Z_i = \left( \bar{R}_i - R_F \right)/\sigma_i (1 - \phi) / \sigma_i = \delta_i \mu_i
\]

Suppose that \( \mu_i > 0 \) in a feasible solution to (1)–(4). By the complementarity condition (3), it follows that \( Z_i = 0 \) and \( \delta_i = 0 \); hence the term in the brackets must be negative. If on the other hand, the term in the brackets is negative, it follows from the nonnegativity of \( Z_i \) and \( \delta_i \) that \( \mu_i > 0 \). By negation, security \( i \) is included in the optimal portfolio if and only if
\[(\bar{R}_i - R_j)/\sigma_i \geq \phi.\] We now prove the following property of the optimal portfolio.

**(P1)** If \((\bar{R}_i - R_j)/\sigma_i \geq (\bar{R}_j - R_j)/\sigma_j\) and security \(j\) is in the optimal portfolio, then security \(i\) is in the optimal portfolio as well.

Indeed, if security \(i\) were not in the optimal portfolio, then from (3) \(\delta_i = 0\) and consequently, for every feasible solution to (1)-(4), \((\bar{R}_i - R_j)/\sigma_i - \phi \geq (\bar{R}_j - R_j)/\sigma_j - \phi > 0\). Thus by our observations preceding (P1), \(Z_i > 0\) in every feasible solution to (1)-(4), contradicting our assumption. Hence, (P1) follows.

Thus one implication of the Kuhn-Tucker conditions is that all securities having \((\bar{R}_i - R_j)/\sigma_i\) greater than the lowest ratio \((\bar{R}_j - R_j)/\sigma_j\) in the optimal portfolio are to be included in the optimal portfolio, whereas all securities whose ratio of excess return to standard deviation is less than \((\bar{R}_j - R_j)/\sigma_j\) are to be excluded from the optimal portfolio. Two problems remain: (1) to find the critical ratio \((\bar{R}_j - R_j)/\sigma_j\), and (2) to determine the amounts to be invested in each security. For the remaining part of this section we assume that all securities have been ordered such that \(i > j\) implies that \((\bar{R}_i - R_j)/\sigma_j \geq (\bar{R}_i - R_j)/\sigma_i\).

Let \(N = \{1, \ldots, n\}\) and assume without loss of generality that \(K \subseteq N\) is the set of securities that are (at positive level) in the optimum portfolio and let \(\kappa = |K|\). Denote by \(H\) the subset of securities in the optimum portfolio that are invested in at their respective upper bounds and let \(\chi = |H|\) be the number of such securities. We show next that the optimal sets \(H\) and \(K\) can be determined by a simple recursive procedure. To do so, we first derive two inequalities that determine \(H\) and \(K\). From (1) we eliminate the expression \(\sum_{j \in K} \sigma_j Z_j\) by multiplying (1) by \(\sigma_i\) and summing over the set \(K\). Noting that \(Z_i = 0\) for \(i \notin K\) and that \(\mu_i = 0\) for \(i \in K\), we obtain

\[
\sum_{i \in K} \sigma_j Z_j = (1 + (\kappa - 1)\rho)^{-1}[\sum_{j \in K} (\bar{R}_j - R_j)/\sigma_j - (1 - \rho) \sum_{j \in K} \sigma_j \delta_j].
\]

Substituting (6) into (1) yields for \(i = 1, \ldots, n\)

\[
Z_i = A_i/(\sigma_i(1 - \rho) + (\rho/\sigma_i)[1 + (\kappa - 1)\rho]) \sum_{j \in K} \sigma_j \delta_j - \delta_i + \mu_i,
\]

where we have set

\[
A_i = (\bar{R}_i - R_j)/\sigma_i - (\rho/[1 + (\kappa - 1)\rho]) \sum_{j \in K} (\bar{R}_j - R_j)/\sigma_j.
\]

Note that \(A_i\) corresponds up to a factor, to the solution of the optimal portfolio selection problem without upper bounds [6]. Let \(H\) be defined as above, i.e., as the subset of those securities in the optimal portfolio for which \(\delta_i > 0\) in a feasible solution to (1)-(4). By the complementarity
condition (3), it follows that
\[ Z_i = C_i \sum_{j \in K} Z_j \quad \text{for all } i \in H. \quad (9) \]

Assume that \( H \neq K \). (This assumption can always be met.) Consequently, \( \sum_{i \in H} C_i < 1 \), since \( Z_j > 0 \) for \( j \in K - H \). Adding all equations (9) we find that every feasible solution to (1)-(4) for which \( H \neq K \) satisfies
\[ Z_i = |C_i| \left( 1 - \sum_{j \in H} C_j \right) \sum_{j \in K - H} Z_j \quad \text{for all } i \in H. \quad (10) \]

Observing that \( \mu_i = 0 \) and \( \delta_i = 0 \) for all \( i \in K - H \), we obtain by adding the appropriate equations in (7) that
\[ \sum_{i \in K - H} Z_i = (1 - \rho)^{-1} \sum_{i \in K - H} A_j / \sigma_j \]
\[ + \left( \rho / (1 + (\kappa - 1) \rho) \right) \sum_{i \in K - H} \sigma_j^{-1} \sum_{j \in K} \sigma_j \delta_j. \quad (11) \]

On the other hand, equating (10) and (7) for \( i \in H \), we obtain, by multiplying the resulting equation by \( \sigma_i \) and then adding,
\[ \sum_{i \in H} \sigma_i C_i (\sum_{i \in H} Z_i) / (1 - \sum_{i \in H} C_i) = (1 - \rho)^{-1} \sum_{i \in H} A_j \]
\[ + \left( \rho / (1 + (\kappa - 1) \rho) \right) \sum_{i \in H} \sigma_i \delta_j. \quad (12) \]

since \( \delta_j = 0 \) for \( j \in K - H \). Solving (11) and (12) for the term \( \sum_{j \in K} \sigma_j \delta_j \), we find that the quantity \( \Phi(H, K) = [\rho (1 - \rho) / (1 + (\kappa - 1) \rho)] \sum_{i \in H} \sigma_i \delta_j \) is given by
\[ \Phi(H, K) = (\rho / \alpha_n) [(1 - \sum_{j \in H} C_j) \sum_{j \in H} A_j \]
\[ - \left( \sum_{j \in H} \sigma_j C_j \right) \sum_{j \in K - H} A_j / \sigma_j], \quad (13) \]

where
\[ \alpha_n = (1 + (\kappa - 1) \rho) \sum_{i \in H} C_i \]
\[ + \rho \left( \sum_{j \in H} \sigma_j C_j \right) \left( \sum_{j \in K - H} \sigma_j^{-1} \right). \quad (14) \]

Equation (7) can now be written as
\[ Z_i = \left( 1 / [\sigma_i (1 - \rho)] \right) [A_i + \Phi(H, K)] - \delta_i + \mu_i. \quad (7') \]

Defining \( \psi(H, K) \) by \( \psi(H, K) = [(1 - \rho) / (1 - \sum_{j \in H} C_j)] \sum_{j \in K - H} Z_j \), we obtain from (11) and (12)
\[ \psi(H, K) = \alpha_n^{-1} [(1 + (\kappa - 1) \rho) \sum_{j \in K - H} A_j / \sigma_j \]
\[ + \rho \left( \sum_{j \in K - H} \sigma_j^{-1} \right) \left( \sum_{i \in H} A_j \right)]. \quad (15) \]

From (9), (10) and the definition of \( \psi(H, K) \), it follows that \( \psi(H, K) = (1 - \rho) \sum_{j \in K} Z_j \) for every feasible solution to (1)-(4) satisfying \( H \neq K \). From the condition that \( Z_i \leq C_i \sum_{j \in H} Z_j = [C_i / (1 - \rho)] \psi(H, K) \), we find for \( i \in K - H \) using (7') with \( \mu_i = 0 \) and \( \delta_i = 0 \) that
\[ (1 / \sigma_i C_i) [A_i + \Phi(H, K)] \leq \psi(H, K) \quad \text{for all } i \in K - H. \quad (16) \]
Similarly, for the set \( H_i = C_i \sum_{j \in K} Z_{ij} = \left[ C_i / (1 - \rho) \right] \psi(H, K) \) for all \( i \in H \).

Using relation (7') to eliminate the variables \( Z_{ij} \) for \( i \in H \), we find from the condition \( \delta_i > 0 \) for \( i \in H \) that

\[
(1/\sigma, C_i) [A_i + \Phi(H, K)] \geq \psi(H, K) \quad \text{for all } i \in H, \quad (17)
\]

since by the complementarity condition \( z_i = 0 \) for \( i \in H \). The quantities \( A_i \) have the general form of the solution to the portfolio selection problem \((P')\) without upper bounds. Since the upper bounds force additional securities to enter the optimal portfolio, the \( A_i \)'s associated with the lower ranking securities must be negative, while \( A_i + \Phi(H, K) \) must be positive. Hence \( \Phi(H, K) \) can be thought of as an "adjustment" to the most negative deviation of the standardized excess return \((\tilde{R}_i - R_i)/\sigma_i\), from the "average standardized excess return" \((\rho/[1 + (\kappa - 1)\rho]) \sum_{i \in K} (\tilde{R}_i - R_i)/\sigma_i\), that results from the upper bounds. Since \( \Phi(H, K) \) is a function of \( \sum_{i \in H} \sigma_i \delta_i \) and since \( \delta_i \)'s are the dual variables associated with the upper bound constraints, \( \Phi(H, K) \) is related to the opportunity cost associated with the upper-bound constraints. Furthermore, securities are selected for investment at their respective upper bounds when their "adjusted" standardized excess return \( A_i + \Phi(H, K) \) divided by their total risk \( \sigma_i C_i \), incurred from investment at the upper bound \( C_i \), exceeds the cut-off value \( \psi(H, K) \). It is important to note that \( \psi(H, K) \) is determined with respect to the adjusted standardized excess return per unit of total risk at the upper bound \((A_i + \Phi(H, K))/\sigma_i C_i\), rather than with respect to \( A_i + \Phi(H, K) \) alone. As a consequence it is possible that a security may be in the optimal portfolio at its upper bound and have a standardized excess-return less than the standardized excess-return of another security in the optimal portfolio that is invested in below its upper bound. This means that the single ordering of the securities given by the decreasing standardized excess return is no longer sufficient to determine the optimal portfolio. Relations (16) and (17) nevertheless permit one to define a simple recursive procedure that determines the optimal set \( H \) and \( K \), respectively. In fact, all that is needed to find an optimal portfolio is a "ranking within the ranking" in order to find out which securities are to be invested in at upper bounds and which are not. The procedure is:

1. (Initialization): Let \( k_0 \) be the smallest index such that \( \sum_{j=1}^{k_0-1} C_j < 1 \) and \( \sum_{j=k_0}^{n} C_j \geq 1 \). Set \( K = \{1, \ldots, k_0\} \) and \( k = k_0 \) and go to 2.
2. Compute \( A_i \) for \( i \in K \) by (8). Set \( L = \max(0, -A_k) \). Go to 3.
3. (Forward Step): If \( A_{k+1} + L \geq 0 \), where \( A_{k+1} \) is computed by (8), redefine \( K \) to be \( K \cup \{k+1\} \) and go to 2. Otherwise, go to 4.
4. (Ranking Step): Order the quantities \((1/\sigma_i C_i)(A_i + L)\) in decreasing order and determine the smallest positive \( \Delta \) such that the order does not change (see the determination of critical values below). Let \( K = \{i_1, \ldots, i_{\Delta}\} \), where the indices have been arranged in the order
just found and determine $k_1$ such that $\sum_{j=1}^{k_1} C_{ij} < 1$, $\sum_{j=1}^{k_1+1} C_{ij} \geq 1$. Go to 5.

5. (Bounding Step): Set $H_j = \{ i_0, \cdots, i_j \}$ for $j = 0, 1, \cdots, k_1$ and compute $\Phi(K) = \max \{ \Phi(H_j, K) | j = 0, 1, \cdots, k_1 \}$ using (13), where $H_0 = \emptyset$ and $\Phi(H_0, K) = 0$. If $\Phi(K) > L+\Delta$, redefine $L$ to be $\Phi(K)$ and go to 3. Otherwise, let $j_0$ be such that $\Phi(K) = \Phi(H_{j_0}, K)$ and compute $\psi(H_{j_0}, K)$ using (15). If (16) and (17) are satisfied for the current $K$ and $H = H_{j_0}$, go to 6. Otherwise, redefine $L$ to be $L+\Delta$ and go to 3.

6. (Termination): If $A_{k+1} \leq -\Phi(K)$, stop. Otherwise, redefine $K$ to be $K \cup \{ k+1 \}$ and go to 2.

In the initial step we determine the minimum number of securities that must be in the portfolio in order to permit full investment. The effect of the next two steps is to ensure that all those securities are included in the optimum portfolio that either would be included if no upper bounds were present or must be considered for inclusion because of property (P1). In the next step securities currently considered for inclusion in the optimum portfolio are ranked in such a way that the order does not change over some finite interval $[L, L+\Delta]$, where $\Delta$, the critical value, is computed below and $L$ is a valid lower bound on the "true," but unknown, quantity $\Phi(H, K)$ appearing in (16) and (17). Also, the maximum number $k_1$ of securities that may be invested in at their respective upper bounds is found. In Step 5 we use the fact proven in the appendix that each one of the quantities $\Phi(H_j, K)$ for $j = 0, 1, \cdots, k_1$ constitutes a valid lower bound on the "true" quantity $\Phi(H, K)$. If $\Phi(K)$, the maximum of all $\Phi(H_j, K)$, exceeds $L+\Delta$, then the rank order must be changed and also, possibly, the portfolio be enlarged. Hence, we return to Step 3. Otherwise, because of (16) and (17) we proceed to compute the quantity $\psi(H_{j_0}, K)$ (or more precisely, if there are ties in the determination of $\Phi(K)$, all those quantities $\psi(H_j, K)$ for which $\Phi(H_j, K) = \Phi(K)$) and check whether $\psi(H_{j_0}, K)$ constitutes the desired cut-off number. If yes, check whether or not the "next" security (next in the order determined by decreasing excess return to standard deviation) needs to be included and proceed accordingly, iterating the whole procedure if necessary. If the answer is no, increase the lower bound to $L+\Delta$ and iterate, returning first to Step 3 and then determining the "new" finite interval for which a new rank order prevails.

The validity of the proposed procedure rests on two assumptions: First, we assume the existence of a solution to problem (P). This, however, is implied by the blanket assumption mentioned in the introduction. Second, we assume the bounding calculations carried out in Step 5 are valid. The validity of this assumption is established in the appendix using linear programming duality. With these two assumptions met, the finite convergence follows from two facts:
(1) The ranking of the quantities \((1/\sigma_i C_i) (A_i + L)\) in Step 4, where \(L\) is regarded as some real parameter, provides a unique partitioning of the real line into a finite number of mutually exclusive intervals, each of which is characterized by a particular ordering; (2) Starting the procedure at some valid lower bound of the "true" \(\Phi(H, K)\) and increasing the "proxy" \(L\) for \(\Phi(H, K)\) monotonically (see Step 5), we must, since \(\Phi(H, K)\) exists, end up after a finite number of steps with the interval into which the "true" \(\Phi(H, K)\) falls.

Before turning to a numerical example, we need to discuss how the critical value \(\Delta\) is determined in Step 4. Suppose that for the current value of \(L\) and \(K\) the quantities \((A_i + L)/\sigma_i C_i\) are ordered in decreasing order of magnitude, i.e., \((A_1 + L)/\sigma_1 C_1 \geq \cdots \geq (A_k + L)/\sigma_k C_k\). If we consider a small positive change \(\epsilon > 0\) in \(L\), this order will generally change.

Setting
\[
\Delta_{ij} = [(A_i + L)/(\sigma_i C_i) - (A_j + L)/(\sigma_j C_j)] / (C_i \sigma_i - C_j \sigma_j) \tag{18}
\]
if \(i < j, C_i \sigma_i > C_j \sigma_j\) and \((A_i + L)/\sigma_i C_i > (A_j + L)/\sigma_j C_i\); and defining \(\Delta_i = \infty\) otherwise, one can readily verify that \(\Delta = \min \Delta_{ij}\). Unless ties arise in (18), it is sufficient to consider only adjacent pairs of indices \((i, i+1)\).

Finally, once the sets \(H\) and \(K\) have been identified, the optimal values of \(Z_i\) are from (7), (9) and (15):

\[
Z_i = (1/\int_{1-\rho}) (A_i + \Phi(H, K)) \text{ for } i \in K - H
\]
\[
Z_i = [C_i/(1-\rho)] \Phi(H, K) \text{ for } i \in H, \quad Z_i = 0 \text{ for } i \in N - K. \tag{19}
\]

In order to obtain the proportions to be invested in the optimal portfolio, we scale the values of \(Z_i\) so that they add to one and obtain

\[
X_i = \begin{cases} 
C_i & \text{for } i \in H \\
(\Phi(H, K))/\sigma_i \Phi(H, K) & \text{for } i \in K - H \\
0 & \text{for } i \in N - K.
\end{cases} \tag{20}
\]

**Example 1.** Consider the data for an optimal portfolio selection problem displayed in Table I and assume that \(\rho = 1/2\). It follows that at least five securities must be in the optimal portfolio. As the securities have been ordered already by decreasing ratio of excess return to standard deviation, we can initialize \(K = [1, 2, 3, 4, 5]\) and compute the \(A_i\) for \(i \in K\) by (8). Carrying out this calculation yields \(A_i = 1, \cdots, 5\), as displayed in Table II. Consequently, \(L = 5/6\). As \(A_i + L = -1\), we do not enlarge the portfolio at this point. Instead, we order (Step 4) the quantities \(D_i = (1/\sigma_i C_i) (A_i + L)\) in decreasing order (see Table II) and find \(K = [2, 3, 1, 4, 5]\). Carrying out the calculations of \(\Delta\), we find that \(\Delta = 2\). From Step 5 we find that \(\Phi(K) = 32/21\) is assumed for \(H = [2, 3, 1, 4]\). As \(\Phi(K) < L + \Delta\), we compute \(\Phi(H, K) = 30/7\). We check that (16) and (17) are in fact satisfied. Since \(A_4 = -5/3 < -32/21\), it follows that \(H = [2, 3, 1, 4]\).
TABLE I

<table>
<thead>
<tr>
<th>No.</th>
<th>$a_i$</th>
<th>$(R_i - R_f)/a_i$</th>
<th>$C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>10</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>9</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>8</td>
<td>$\frac{1}{10}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{2}$</td>
<td>7</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>6</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1}{2}$</td>
<td>5</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>5</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>5</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>5</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>5</td>
<td>$\frac{1}{6}$</td>
</tr>
</tbody>
</table>

and $K = \{2, 3, 1, 4, 5\}$ are optimal. Using (20) we find that the fractions in the optimal portfolio are $X_1 = X_2 = \frac{1}{6}$, $X_3 = X_5 = 1/10$ and $X_4 = \frac{1}{6}$.

3. THE SINGLE-INDEX MODEL AND THE CONSTRUCTION OF OPTIMAL PORTFOLIOS

In this section we determine optimal procedures for portfolio selection under the assumptions of the standard single-index model. Since the arguments in this section are analogous to those of Section 2, they will be presented more succinctly. The single-index model can be written as:

1. $R_i = \alpha_i + \beta_i I + \epsilon_i$, $i = 1, \cdots, N$
2. $I = \alpha_{N+1} + \epsilon_{N+1}$
3. $E(\epsilon_{N+1}\epsilon_i) = 0$, $i = 1, \cdots, N$
4. $E(\epsilon_i \epsilon_j) = 0$, $i, j = 1, \cdots, N$, $i \neq j$

where $R_i$ is the return on security $i$, a random variable; $I$ is a market index, a random variable with mean $\alpha_{N+1}$ and variance $\sigma^2$; $\beta_i$ is a measure of the responsiveness of security $i$ to changes in the market index; $\alpha_i$ is the return on security $i$ that is independent of changes in the market index;

TABLE II

<table>
<thead>
<tr>
<th>$i = {1, 2, 3, 4, 5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_i$</td>
</tr>
<tr>
<td>1%</td>
</tr>
<tr>
<td>3%</td>
</tr>
<tr>
<td>3%</td>
</tr>
<tr>
<td>1%</td>
</tr>
<tr>
<td>-3%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$i = {1, 2, 3, 4, 5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_i$</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>15</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$
\( \epsilon_i \) is a variable with a mean of zero and variance \( \sigma_i^2 \) for \( i = 1, \ldots, N; \epsilon_{N+1} \) is a random variable with mean of zero and variance \( \tau^2 \).

We will assume throughout this section that all \( \sigma_i^2 \neq 0 \) and \( \beta_i \neq 0 \). Furthermore, reflecting the economic meaning of the \( \beta_i \)'s, we can assume that "almost all" \( \beta_i \)'s are positive and that negative \( \beta_i \)'s are not "too large" in absolute value (see appendix). Finally, we will assume that it is possible to satisfy the upper bound constraints using only stocks with positive \( \beta_i \)'s.

Using similar (positivity-preserving) transformations as in Section 2, the Kuhn-Tucker conditions for the maximization problem (P) become

\[
Z_i = \frac{(\bar{R}_i - R_f)}{\sigma_i^2} - \beta_i \sum_{j=1}^{n} \beta_j Z_j - \delta_i + \mu_i \tag{21}
\]

\[
Z_i \leq C_i \sum_{j=1}^{n} Z_j \tag{22}
\]

\[
Z_i \mu_i = 0, \quad (Z_i - C_i \sum_{j=1}^{n} Z_j) \delta_i = 0 \tag{23}
\]

\[
Z_i \geq 0, \quad \mu_i \geq 0, \quad \delta_i \geq 0. \tag{24}
\]

Setting \( \phi = \tau^2 \sum_{j=1}^{n} \beta_j Z_j \), (21) becomes

\[
Z_i = \beta_i \sigma_i^{-2} \left( \frac{(\bar{R}_i - R_f)}{\beta_i - \phi} \right) - \delta_i + \mu_i. \tag{25}
\]

By an argument analogous to the one used in connection with (P1) we have

(P2) If \( (\bar{R}_i - R_f)/\beta_i \geq (\bar{R}_k - R_f)/\beta_k \), \( \beta_i > 0 \), \( \beta_k > 0 \) and security \( k \) is in the optimal portfolio, then so is security \( i \). If \( (\bar{R}_i - R_f)/\beta_i \leq (\bar{R}_k - R_f)/\beta_k \), \( \beta_i < 0 \), \( \beta_k < 0 \) and security \( k \) is in the optimal portfolio, then so is security \( i \).

Thus there exists a stock \( k \) among all stocks with positive (negative) \( \beta_i \)'s such that all stocks having a ratio of excess return to beta greater (less) than \( (\bar{R}_k - R_f)/\beta_k \) are included in the optimal portfolio.

Let \( N = \{1, \ldots, n\} \) be the set of all stocks considered, \( K \subset N \) be the set of stocks in an optimal portfolio, and \( H \subset K \) be the set of stocks that are at upper bounds in an optimal portfolio. Thus, \( i \not\in H \) implies \( \delta_i > 0 \) for a feasible solution to (21)-(24). Define \( B_i \) as

\[
B_i = (\bar{R}_i - R_f)/\beta_i - \gamma \sum_{j \in K} \beta_j (\bar{R}_j - R_f)/\sigma_j^2, \tag{26}
\]

where \( \gamma = \tau^2/(1 + \tau^2 \sum_{j \in K} (\beta_j^2/\sigma_j^2)) \). We use a procedure parallel to the one used in Section 2 to compute the quantities \( \Phi(H, K) \) and \( \psi(H, K) \). Corresponding to (7) we find, employing (21), that

\[
Z_i = (\beta_i/\sigma_i^2) B_i + \gamma (\beta_i/\sigma_i^2) \sum_{j \in K} \beta_j \delta_j - \delta_i + \mu_i \tag{27}
\]

Corresponding to (13) and (14), \( \Phi(H, K) = \gamma \sum_{i \in H} \beta_i \delta_i \) is given by

\[
\Phi(H, K) = \gamma \left( \frac{\tau^2}{\alpha N} \right) \left[ (1 - \sum_{j \in H} C_j) \sum_{i \in H} (\beta_i^2/\sigma_i^2) B_i - (\sum_{j \in H} \beta_j C_j) \sum_{i \in K - H} (\beta_i/\sigma_i^2) B_i \right] \tag{28}
\]
where
\[ \alpha_N = \left[ 1 + \tau^2 \sum_{j \in K-H} (\beta_j^2/\sigma_j^2) \right] \left( 1 - \sum_{j \in H} C_{ij} \right) \]
\[ + \tau^2 \left( \sum_{j \in H} \beta_j C_{ij} \right) \sum_{i \in K-H} \left( \beta_i/\sigma_i^2 \right). \]  
(29)

Setting \( \psi(H, K) = \left[ 1 - \left( 1 - \sum_{i \in H} C_{ij} \right) \right] \sum_{i \in K-H} Z_{ki} \), we obtain
\[ \psi(H, K) = \alpha_N^{-1} \left\{ \tau^2 \sum_{i \in K-H} (\beta_i/\sigma_i^2) \sum_{j \in H} (\beta_j^2/\sigma_j^2) B_{ij} \right. \]
\[ + \left. (1 + \tau^2 \sum_{i \in K-H} (\beta_i^2/\sigma_i^2)) \sum_{i \in K-H} (\beta_i/\sigma_i^2) B_{ij} \right\} \]  
(30)

Employing the condition that \( Z_i \leq C_i \sum_{j \in K} Z_j \) for all \( i \in K-H \), we get from (27) with \( \delta_i = 0 \) and the respective definitions of \( \Phi(H, K) \) and \( \psi(H, K) \)
\[ (\beta_i/\sigma_i^2 C_i) [B_i + \Phi(H, K)] \leq \psi(H, K) \]  
for all \( i \in K-H \).  
(31)

Expressing the conditions that \( \delta_i \geq 0 \) for \( i \in H \), we get from (27) and
\[ Z_i = C_i \psi(H, K) \]  
for \( i \in H \), the reverse inequality
\[ (\beta_i/\sigma_i^2 C_i) [B_i + \Phi(H, K)] \geq \psi(H, K) \]  
for all \( i \in H \).  
(32)

Hence we can again define a recursive procedure to identify both the set \( H \) and \( K \) if we know a systematic way of "underestimating" the "true" \( \Phi(H, K) \). If all \( \beta \)'s are positive, the procedure of Section 2 produces \textit{mutatis mutandis} optimal sets \( H \) and \( K \). Note that the necessary changes involve replacing the term \( \sigma_i^2/\beta_i \) by the term \( (1-\rho)\sigma_i \) (see the appendix). In order to compute the optimal solution in terms of the \( Z_i \), we need the following formulas, which correspond to (19):
\[ Z_i = (\beta_i/\sigma_i^2) [B_i + \Phi(H, K)] \]  
for \( i \in K-H \)
\[ Z_i = C_i \psi(H, K) \]  
for \( i \in H \)
\[ Z_i = 0 \]  
for \( i \in N-K \)  
(33)

The proportions for the optimal portfolio are then obtained by scaling the \( Z_i \) to add to one.

If negative \( \beta \)'s are present, the procedure for calculating the \textit{critical value} \( \Delta \) must be modified slightly. To ensure convergence, it is necessary to check that the "feasibility" conditions (A-8) of the appendix are met whenever negative \( \beta \) stocks are considered for inclusion. Since negative \( \beta \) stocks are rarely found, we propose to circumvent these technical difficulties by first solving the portfolio problem for positive \( \beta \) stocks and then checking if any of the negative \( \beta \) stocks need be included.

4. CONCLUSION

In this paper we develop a new procedure for optimal portfolio selection with upper bounds for the two more frequently applied models in portfolio selection. Extending our previous work [5, 6], we show that upper bounds
can be dealt with in a slightly more complex fashion that, however, still shares many of the features of the ranking procedures of [5, 6]. In fact, portfolio selection with upper bounds involves two steps: (1) a pre-ranking of stocks identical to the one used in [5, 6], which determines the order in which stocks enter into the optimal portfolio; and (2) a further ranking that determines which of the selected stocks are to be held at their upper bounds. Though our limited computational experience indicates that the latter ranking is a one-shot operation, this need not necessarily be so and it may be necessary to iterate.

An obvious alternative approach to selecting optimal portfolios with upper bounds is the “piece-meal” approach, which first determines an optimal portfolio without upper bounds, then fixes the most preferred stock at its respective upper bound (if binding), and then iterates by distributing the remaining wealth among the remaining stocks. Though very practical, this method produces portfolios that generally are not optimal. This can be seen by carrying out the procedure for the numerical example of Section 2.

ACKNOWLEDGMENT

We would like to thank David Baron for helpful comments.

APPENDIX

We prove that the bounding calculations performed in Step 5 of the procedure described in Section 2 are in fact valid. Note that because of the assumed positivity of $\rho$, the system (1)-(4) is formally subsumed by (21)-(24). This can be seen by replacing $\sigma^2$ by $\rho$, $\beta_i$ by $\sigma_i$, and $\sigma^2_i$ by $(1-\rho)\sigma_i^2$. Hence we need only concern ourselves with the system (21)-(24). Our aim is to establish that by the calculations of Step 5 we always underestimate the “true” quantity $\Phi(H,K)$ for the given problem. Our proof produces usable formulas that enable one to monitor the validity of the proposed procedure even in the case that negative $\beta$ stocks enter the optimal portfolio. These conditions (A-8) are always satisfied if all $\beta$’s are positive, i.e., for the portfolio model of Section 2. In order to establish validity of the bounding calculations, it suffices to do so for a fixed set $K$. For, suppose that for a given set $K$ the bounding calculations are valid and that $K$ is enlarged by a security $k+1$ (in Step 6 or 3). If $L$ is indeed less than or equal to the “true” $\Phi(H,K)$, then the test carried out in Step 3 or 6 indicates (see relation (5) or (25)) that security $k+1$ must be considered for inclusion in the optimal portfolio. Hence, restarting the lower bound at $L$ as defined in Step 2 definitely is valid. Consequently, we need to consider only the case of a fixed set $K$.

Consider the system (21)-(24) and rewrite the conditions, eliminating
the variables \( Z_i \). Then we get the following system in the unknowns \( \delta_i \), for all \( i \in K \):

\[
-\delta_i + \gamma (\beta_i / \sigma_i^2) \sum_{k \in K} \beta_k \delta_k \geq - \left( \beta_i B_i / \sigma_i^2 \right), \tag{A-1}
\]

\[
\delta_i - C_i \sum_{k \in K} \delta_k - \gamma (\beta_i / \sigma_i^2 - C_i \sum_{k \in K} \beta_k / \sigma_k^2) \sum_{k \in K} \beta_k \delta_k \geq \beta_i B_i / \sigma_i^2 - C_i \sum_{k \in K} \beta_k B_k / \sigma_k^2, \tag{A-2}
\]

\[
\delta_i \geq 0 \tag{A-3}
\]

where \( \gamma = \tau^2 / (1 + \tau^2 \sum_{k \in K} \beta_k^2 / \sigma_k^2) \).

Conditions (A-1) express the nonnegativity of the \( Z_i \), and those of (A-2) correspond to the constraints (22). The complementarity condition (23) now means that \( \delta_i > 0 \) in any feasible solution implies that the \( i \)th inequality of (A-2) must hold with equality. Assume now for simplicity that \( K = \{1, \ldots, m\} \) and let \( \delta_1, \ldots, \delta_m \) be any feasible solution to (A-1)–(A-3) that also satisfies the complementarity condition. As we have mentioned before, we can assume the existence of such a solution. Suppose that \( \delta_1 > 0, \ldots, \delta_i > 0 \), whereas \( \delta_{i+1} = \cdots = \delta_m = 0 \). Solving the first \( h \) equations of (A-2) we can calculate the solution explicitly:

\[
\delta_i = \left( \beta_i / \sigma_i^2 \right) B_i + \alpha(k) \sum_{j=1}^h (\beta_j / \sigma_j^2) B_j - \beta(k) \sum_{j=h+1}^m (\beta_j / \sigma_j^2) B_j \tag{A-4}
\]

for \( k = 1, \ldots, h \), where

\[
\alpha(k) = \frac{1}{\sigma_i} \left[ (\beta_i / \sigma_i^2) \left( 1 - \sum_{j=1}^h C_j \right) + C_k \sum_{j=h+1}^m \beta_j / \sigma_j^2 \right] \tag{A-5}
\]

\[
\beta(k) = \frac{1}{\sigma_i} \left[ C_i (1 + \tau^2 \sum_{j=h+1}^m \beta_j^2 / \sigma_j^2) + \tau^2 (\beta_i / \sigma_i^2) \sum_{j=1}^h \beta_j C_j \right]
\]

and \( \alpha_n \) is defined by (29) with \( H = \{1, \ldots, h\} \). It can be verified by inspection that (A-4), (A-5) in fact constitute a solution to the first \( h \) equations satisfying \( \delta_{h+1} = \cdots = \delta_m = 0 \). Moreover, if \( \alpha_n \neq 0 \), this solution is uniquely defined and hence, by assumption, a basic feasible solution to (A-1)–(A-3). (In order to rederive (A-4) one should note that the matrix to be inverted can be written as \( A - w^T \), where both \( u \) and \( v \) are column vectors, but \( (A - w^T)^{-1} = A^{-1} + [1 / (1 - \sigma^2 A^{-1} w)] (A^{-1} u) (v^T A^{-1}) \). Applying this formula twice yields the desired inverse.) Computing \( \sum_{j=1}^h \beta_j \beta_j \) we find that

\[
\sum_{j=1}^h \beta_j \beta_j = \Phi(H, K) / \gamma, \tag{A-6}
\]

where \( \Phi(H, K) \) is given by (28) with \( H = \{1, \ldots, h\} \). In order to prove the asserted bounding property, consider the following linear programming problem

\[
(DP) \quad \max V(u, v) = \sum_{i=1}^n (-\beta_i B_i / \sigma_i^2) u_i + \sum_{i=1}^m (\beta_i B_i / \sigma_i^2 - C_i \sum_{j=1}^m \beta_j B_j / \sigma_j^2) v_i - u_k + \gamma \beta_k \sum_{i=1}^m (\beta_i / \sigma_i^2) u_i + u_k - \gamma \beta_k \sum_{i=1}^m (\beta_i / \sigma_i^2 - C_i \sum_{j=1}^m \beta_j / \sigma_j^2) v_i
\]

\[
- \sum_{j=1}^m C_j \beta_j \leq \beta_k, \quad k = 1, \ldots, m
\]

\[
u_k \geq 0, \quad v_k \geq 0.
\]
With little algebraic effort one can show that
\[
\bar{\alpha} = 0, \quad i = 1, \ldots, m
\]
\[
\bar{\omega} = \left( r^2 / \gamma \alpha \nu \right) \left[ \beta_k \left( 1 - \sum_{j=1}^{k} C_j \right) + \sum_{j=1}^{k} C_j \beta_j \right], \quad k = 1, \ldots, h \quad (A-7)
\]
\[
\bar{\beta}_k = 0, \quad k = h+1, \ldots, m
\]
is a feasible solution to (DP) provided that
\[
\left( \frac{r^2}{\gamma \alpha \nu} \right) \left[ \beta_k \left( 1 - \sum_{j=1}^{k} C_j \right) + \sum_{j=1}^{k} C_j \beta_j \right] \geq 0 \quad \text{for } k = 1, \ldots, h
\]
\[
-\omega \nu \left( 1 - \sum_{j=1}^{k} C_j \right) \beta_p \leq \omega \nu \sum_{j=1}^{k} C_j \beta_j \quad \text{for } p = h+1, \ldots, m \quad (A-8)
\]
Note that if \( \sum_{j=1}^{h} C_j < 1 \) and all \( \beta_j > 0 \) for \( j = 1, \ldots, m \), the conditions (A-8) are always satisfied. In particular, in this case, \( \alpha \nu > 0 \). Suppose now that \( (\bar{u}, \bar{v}) \) is feasible. Computing the objective function value of (DP) we find that
\[
V(\bar{u}, \bar{v}) = \Phi(H, K) / \gamma.
\]
Consequently, (DP) is the dual linear program to the problem
\[
\text{min } \sum_{j=1}^{n} \beta_j \beta_j \quad \text{subject to } (A-1), (A-2), (A-3).
\]
It follows that the solutions that satisfy (A-1), (A-2), (A-3) and additionally the complementarity condition are among the optimal solutions of (PP). As the set \( H = \{ 1, \ldots, h \} \) is perfectly arbitrary, it follows furthermore that if \( \sum_{j=1}^{h} C_j < 1 \) and \( \beta_j > 0 \) for \( j = 1, \ldots, m \), the vector \( (\bar{u}, \bar{v}) \) given by (A-7) is a feasible solution to (DP). Hence, by linear programming duality \( V(\bar{u}, \bar{v}) \) is a valid lower bound on the optimal value of the objective function of (PP). As we consider sets \( H \) in the procedure satisfying \( \sum_{j=1}^{h} C_j < 1 \), the bounding procedure always produces a valid lower bound on the “true” value \( \Phi(H, K) \). This is so because, by choosing some subset \( H \) of \( K \), one gets a feasible solution for (DP) whose correspondent primal solution as defined by (A-4) need not be feasible for (PP). Indeed, expressing the feasibility conditions that the solution \( \bar{\delta}_k \) given by (A-4) with \( \bar{\delta}_k = 0 \) for \( k \in K - H \) is feasible for (PP), i.e., satisfies \( \bar{\delta}_k \geq 0 \) for \( k \in H \), the constraints (A-1) and those constraints of (A-2) with \( k \in K - H \) produce exactly the conditions (31) and (32) of Section 3. By construction, of course, the inequality with \( k \in H \) will always be satisfied with equality. The task of checking the remaining inequalities and of verifying that they indeed reproduce (31) and (32) is left to the reader.

Returning to the discussion of negative \( \beta \) stocks, we infer from (A-8) that as long as negative \( \beta \) stocks do not enter into the set \( \bar{K} \) (of stocks considered for investment) their presence in no way influences the question of the validity of the procedure of Section 3. Interpreting the second of the two conditions (A-8) we find that the constraint will in general hold if \( \beta_p < 0 \) is not “too large” in absolute value, i.e., if \( \beta_p \) is sufficiently close to zero the \( p \)th constraint will be satisfied. A similar statement can be made regarding the first set of conditions. Of course, in the most general case of both negative and positive \( \beta \)’s the conditions (A-8) need not be
satisfied and hence, our proposed procedure need not work. But as pointed out previously, from the economic meaning of the $\beta$'s, large negative $\beta$'s can be ruled out with virtual certainty.

REFERENCES


