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Simple Rules for Optimal Portfolio Selection In
Stable Paretian Markets

VIJAY S. BAWA†*, EDWIN J. ELTON* and MARTIN J. GRUBER*

I. Introduction

It is well known that when the joint probability distribution of security returns
is multivariate normal, optimal portfolios for all risk-averse investors lie on the
Markowitz-Tobin mean, variance admissible frontier. The optimal portfolio can
be obtained by matrix inversion (see, for example, Lintner [10] or Merton [12])
if short sales are permitted; otherwise, mathematical programming algorithms
are needed to solve the underlying parametric quadratic programming problem.
Sharpe [16] has shown that if the security returns are generated by a single-index
model and if residuals are uncorrelated, then the underlying optimization problem
is simplified with significant computational savings. In a series of recent papers,
Elton, Gruber and Padberg [3, 4, 5, 6, 7] have shown that under the assumption
of the Sharpe single-index model, as well as for several other models that generate
special covariance structures, optimal portfolios can be constructed by simple
rules that do not involve using quadratic programming techniques. Furthermore,
these rules lead to a unique ranking of securities such that the desirability of any
security for inclusion in a portfolio can be judged before the portfolio composition
itself is obtained. This provides an intuitive explanation to the practitioner as to
why (and how much of) a security should be included in a portfolio.

It is also known (see, for example, Bawa [1], Fama [8, 9], Press [13, 14] and
Samuelson [15]) that when the joint probability distribution of security returns
follow a multivariate stable distribution with characteristic exponent \( \alpha \), \( 0 < \alpha \leq 2 \), portfolio returns follow a univariate stable distribution with the same
characteristic exponent \( \alpha \). In addition, the distributions of portfolio returns belong to a
two parameter (location, scale) family of distributions for (1) Symmetric nonnor-
mal multivariate stable distributions as defined by Press [13, 14], (2) A linear
dependence structure with nonnormal multivariate stable distributions as defined
by Fama [8, 9] or Samuelson [16] for the symmetric case and Bawa [2] for the
asymmetric case. For these distributions, mean is the appropriate return measure
while scale parameter (also referred to as dispersion parameter in the literature)
is the appropriate risk measure. Thus for an arbitrary value of \( \alpha \), \( 1 < \alpha \leq 2 \),
optimal portfolios can be obtained using mathematical programming techniques
(see, for example, Ziemba [16]) that are very similar, though not identical to the

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1 It should be noted that the asymmetric stable distributions defined in Bawa [2] are generalizations
of symmetric stable distributions in [8], [9] and [15], with the skewness parameter \( \beta \) assumed to an
arbitrary constant, \(-1 \leq \beta \leq 1\), across security returns. \( \beta = 0 \) corresponds to symmetric distributions.

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ones employed for the case of normal distributions.\footnote{It is known that diversification is preferred by all risk-averse investors only for \(1 < \alpha \leq 2\). For \(\alpha \leq 1\), specialization is the preferred strategy. As in \([8],[9]\) and \([15]\), we assume that \(1 < \alpha \leq 2\).} The scale parameter of the portfolio replaces the standard deviation used in the case of normal distribution and the underlying optimization problem being solved is a parametric convex programming problem, instead of a parametric quadratic programming problem.

In this paper we show that optimal portfolios can be obtained by simple procedures for multivariate stable distributions with arbitrary \(1 < \alpha \leq 2\), under several specific models of the dependence structure including a single-index model. Our assumptions are analogous to those in Elton, Gruber, and Padberg \([3]\) and the simple procedures are analogous to, though slightly more complex than those developed in \([3]\) for mean variance portfolio problems. Of particular interest is the fact that there is an optimal way to rank securities for inclusion in the optimal portfolio for arbitrary \(1 < \alpha \leq 2\) that is analogous to that which was proved optimal for the mean variance case \([3]\). Our results provide simple decision rules to solve the portfolio problems posed in Fama \([8]\), the single-index model case in Samuelson \([15]\) and Bawa \([2]\), and for special dependence structure cases in Press \([13, 14]\). They also provide an intuitive explanation to the practitioner as to why (and how much of) a security should be included in the optimal portfolio and constitute an important step in actual implementation of portfolio theory.

In the paper, we will only consider assumptions analogous to those made in \([3]\), that is (i) no short sales are allowed, (ii) a riskless asset exists and one can borrow and lend any amount of the riskless asset and (iii) the joint probability distribution of security returns is multivariate stable with characteristic exponent \(\alpha, 1 < \alpha \leq 2\), defined either by a single-index model or the special case of Press’s \([13, 14]\) model with constant coscale factor (analogue of constant correlation for normal distributions). Since most institutions are either prevented by law from short selling or choose not to do so, this is a particularly relevant case. In Sections 2 and 3 we consider the case of a single index model generating security returns and Press’s model with constant coscale factor respectively and show that all the results obtained in \([3]\) carry over. Note that in going from the normal distribution to the nonnormal stable family, the solution to the underlying optimization involves algebraic equations of order \((\alpha - 1)\) (where \(\alpha\) is the characteristic exponent, \(1 < \alpha \leq 2\)) rather than linear (i.e., when \(\alpha = 2\)) equations. Thus all of the results obtained in \([4]-[7]\) carry over from normal distributions to the entire family of stable distributions except for appropriate notational changes and associated higher degree of algebraic complexity and manipulations in deriving the results. For example, those who may be bothered by the assumption of a riskless asset can consider it a mathematical artifact and can use the results of this paper to solve for the full admissible frontier by varying the riskless rate (see Elton, Gruber, and Padberg \([4]\)).

II. Optimal Portfolios for Single Index Model

We let \(R_i\) denote the random return on security \(i, i = 1, 2, \ldots, n\). We assume that...
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$R_i$, with $R' = (R_1, R_2, \ldots, R_n)$, has a multivariate stable distribution with characteristic exponent $\alpha$, $1 < \alpha \leq 2$ and constant skewness index for each security. We further assume that the comovement of security returns is given by the following single index model:

$$R_i = \alpha_i + \beta_i I + \epsilon_i \quad i = 1, 2, \ldots, n$$

(1)

where $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, I)$ are assumed to be independent random variables with univariate stable distributions with common characteristic exponent $\alpha$ and a common skewness index. In addition, we let

- $\mu_m = E(I)$
- $\beta_i$ = scale parameter of the market index $I$
- $\alpha_i$ = a measure of responsiveness of security $i$ to changes in the market index.
- $\epsilon_i$ = return on security $i$ that is independent of changes in the market index.
- $\epsilon_i$ = a random variable with mean zero and scale parameter $S_i$.

The assumption implied by the index model is that the only joint movement between securities comes because of a common response to a market index.

If we let $X_i$ denote the fractional amount invested in security $i$, $i = 1, 2, \ldots, n$, let $R_X$ denote the return on portfolio $X$, and define $\bar{R}_i = E(R_i)$, then with no short selling,

$$\bar{R}_X = E(R_X) = \sum_i X_i \bar{R}_i$$

(2)

and$^{3, 4}$

$$S_X = \text{scale parameter of portfolio } X,$$

$$S_X = \left( \sum_i X_i^\alpha S_i^\alpha + S_m^\alpha \left( \sum_i \beta_i X_i \right)^\alpha \right)^{1/\alpha}$$

(3)

If we let $R_F$ denote the return on a riskless asset, then it follows from the well known separation result (see, for example, Fama [9], Ziemba [17] for a proof for symmetric stable distributions and Bawa [2] for asymmetric stable distributions) that all risk-averse investors will allocate the wealth between the riskless asset and a mutual fund of risky assets. The composition of this mutual fund is obtained via the solution to the following optimization problem:

$$\max \theta = \frac{\bar{R}_X - R_F}{S_X}$$

(4)

subject to

$$\sum_i X_i = 1$$

$$X_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n$$

$^3$Note that for symmetric distributions, (3) has been derived in Fama [8, 9] and Samuelson [15]. For asymmetric distributions with constant skewness index, Bawa [2] has shown that the same result holds.

$^4$We assume $\beta_i > 0$. This is the only empirically relevant case.
We note that \( \theta \) is homogeneous of degree zero and the optimal solution to (1) can be obtained by first maximizing \( \theta \) subject to only the nonnegativity constraints and then scaling the optimal solution. Substituting for \( \bar{R}_X \) and \( S_x \) from (2) and (3), we note that the Kuhn-Tucker conditions for the optimal solution subject to only nonnegativity constraints are:

\[
(R_i - R_F)S_x^{-1} - (R_X - R_F).
\]

\[
(S^\alpha_i X_i^{-1} + S^\alpha_m (\sum_{j=1}^n X_j \beta_j)^{\alpha-1} \beta_i)S_x^{-\alpha} + \mu_i = 0 \quad i = 1, 2, \ldots, n \tag{4}
\]

\[
X_i \mu_i \geq 0 \quad i = 1, 2, \ldots, n \tag{5}
\]

\[
X_i \mu_i = 0 \quad i = 1, 2, \ldots, n \tag{6}
\]

Multiplying through equation (4) by \( S_X \), defining \( Z_j = X_j \left( (R_X - R_F)^{1/\alpha-1}S_X^{-1/\alpha} \right) \) and redefining \( \mu_i = \mu_i S_X S^{-\alpha} \), and rearranging equation (4) yields:

\[
Z_i^{-1} = \frac{(R_i - R_F)}{S^\alpha_i} - \frac{S^\alpha_m (\sum_{j=1}^n Z_j \beta_j)^{\alpha-1} \beta_i}{S^\alpha_i} + \mu_i
\]

or

\[
Z_i^{-1} = \frac{\beta_i}{S^\alpha_i} \left( \frac{R_i - R_F}{\beta_i} - \phi\right) + \mu_i \tag{7}
\]

where

\[
\phi = S^\alpha_m (\sum_{j=1}^n Z_j \beta_j)^{\alpha-1}. \tag{8}
\]

Equation (7) is identical in form to those found in Elton, Gruber and Padberg [3] and all of the conclusions discussed there follow immediately. First if \( \beta_i > 0 \) a security can only be included in the optimum portfolio if the term in parentheses is positive. As discussed in footnote 4 we feel \( \beta_i > 0 \) is the only empirically relevant case. Thus \( \beta_i/S^\alpha_i \) is positive. If \( Z_i > 0 \) then \( \mu_i = 0 \). Thus for \( Z_i \) to be greater than zero the term in the parentheses must be positive.

Secondly, securities can be ranked by their excess mean return to Beta ratios with the optimum portfolio consisting of the top ranked securities. Assume security \( a \) is in the optimum portfolio. Then security \( b \) if it has a higher excess return to Beta ratio must have a larger value for the term in the brackets. Since by assumption the term in the brackets was positive for security \( a \), it must be positive for security \( b \) which has a higher excess return to Beta ratio. Thus the excess return to Beta ratio is a natural ranking device and the highest ranked

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5 Kuhn-Tucker conditions are necessary and sufficient for the optimization problem at hand if one can show that \( \theta \) is a pseudo concave function of \( X \) (Margaarian, O. L. [11; p. 145] shows concavity is not needed—only pseudo concavity is required). It is shown in Margaarian [11; p. 198] that if \( \theta = \phi/\psi \) and \( \phi \) is linear and positive and \( \psi \) is convex and \( \psi > 0 \), then \( \theta \) is pseudo concave. \( \psi = S \) the scale parameter which is positive for all \( X \) and convex. \( \phi = \bar{R}_X - R_F \). For equilibrium \( E(R_i) > R_F \); hence \( \bar{R}_X > R_F \) for all \( X \geq 0 \). Thus Kuhn-Tucker conditions are necessary and sufficient.
securities are included in the optimum portfolio when returns are non-normal stable. This is identical to the conclusion reached in [3] when mean and variance are the appropriate characteristics of portfolios.

Let set \( \kappa \) be the set of securities included in the optimum portfolio. Since \( Z_j = 0 \) for securities not in set \( \kappa \) we have \( \Sigma_{j \in \kappa} Z_j \beta_j = \Sigma_{j \in \kappa} Z_j \beta_j \) and equation (8) becomes \( \phi_\kappa = S_m^2 (\Sigma_{j \in \kappa} Z_j \beta_j)^{\alpha-1} \). Also we have for elements of the set \( \kappa \), \( \mu_j = 0 \). Thus for elements in set \( \kappa \), we have

\[
Z_i^{\alpha-1} = \frac{\beta_i}{S_i^\alpha} \left( \frac{\bar{R}_i - R_F}{\beta_i - \phi_\kappa} \right).
\]  

Taking the \((\alpha - 1)\) root of both sides of (9) and substituting in for \( \phi_\kappa \), we obtain

\[
Z_i = \left( \frac{(\bar{R}_i - R_F)}{S_i^\alpha} - \frac{S_m^2}{S_i^\alpha} \beta_i \beta_j (\Sigma_{j \in \kappa} Z_j \beta_j)^{\alpha-1} \right)^{\frac{1}{\alpha-1}}.
\]

Multiplying by \( \beta_i \) and summing both sides over set \( \kappa \) yields

\[
\Sigma_{j \in \kappa} Z_j \beta_j = \Sigma_{j \in \kappa} \left( \frac{(\bar{R}_i - R_F) \beta_j^{\alpha-1}}{S_i^\alpha} - \frac{S_m^2}{S_i^\alpha} \beta_j^2 (\Sigma_{j \in \kappa} Z_j \beta_j)^{\alpha-1} \right)^{\frac{1}{\alpha-1}},
\]

or

\[
1 = \Sigma_{j \in \kappa} \left( \frac{(\bar{R}_i - R_F) \beta_j^{\alpha-1}}{S_i^\alpha} \cdot \frac{1}{(\Sigma_{j \in \kappa} Z_j \beta_j)^{\alpha-1}} - \frac{S_m^2 \beta_j^2}{S_i^\alpha} \right)^{\frac{1}{\alpha-1}}.
\]

This equation has the form

\[
1 = \Sigma_{j \in \kappa} \left( \frac{D_j}{\phi_j} - \bar{R}_j \right)^{\frac{1}{\alpha-1}}
\]

where \( D_j \) is always positive. While the solution to this equation for \( \phi_\kappa \) initially appears difficult the nature of the problem vastly simplifies the task. From equation (9) we set that if securities are ranked by excess return to Beta and security \( j \) is the lowest ranked security included in the portfolio

\[
\frac{\bar{R}_j - R_F}{\beta_j} \geq \phi_\kappa \quad \text{for} \quad \frac{\bar{R}_{j+1} - R_F}{\beta_{j+1}}.
\]

In addition at the optimum, equation (10) must be satisfied. Notice that for equation (10) to be satisfied when \( j \) is the last security included in the portfolio the RHS of equation (10) must be greater than or equal to one for \( \phi_\kappa = \frac{\bar{R}_{j+1} - R_F}{\beta_{j+1}} \). This is true for the RHS of (10) decreases monotonically with increases in \( \phi_\kappa \). This property together with the fact that the RHS of (10) increases any time a security is added to a portfolio for a given \( \phi_\kappa \) can be used to find the composition of the optimal portfolio.
Thus the optimal set of securities to include can be found by the following straightforward sequential procedure: initially define set \( \kappa \) as containing the first security and let \( \phi_\kappa = \frac{\bar{R}_1 - R_F}{\beta_1} \). If the RHS of (10) is less than one it cannot equal one anywhere in the range \( \frac{\bar{R}_1 - R_F}{\beta_1} \geq \phi_\kappa \geq \frac{\bar{R}_2 - R_F}{\beta_2} \). Next define set \( \kappa \) as including the first two securities and set \( \phi_\kappa = \frac{\bar{R}_3 - R_F}{\beta_3} \). If the RHS of (10) is less than one follow the same procedure with respect to security 3. Continue until with the first \( j \) securities in the included set \( \kappa \) and \( \phi_\kappa = \frac{\bar{R}_{j+1} - R_F}{\beta_{j+1}} \) the RHS of equation (10) is greater than one. The optimum set will then include the first \( j \) securities.

To complete the construction of the optimal portfolio the value of \( \phi_\kappa \) must be determined. Equation (10) must be solved for \( \phi_\kappa \). The solution is facilitated by knowledge that when \( j \) securities are included in the optimal portfolio, \( \phi_\kappa \) must be smaller than \( \frac{\bar{R}_j - R_F}{\beta_j} \) and larger than or equal to \( \frac{\bar{R}_{j+1} - R_F}{\beta_{j+1}} \). Once \( \phi_\kappa \) is determined the value of \( Z_i \) can be determined for all securities with \( \frac{\bar{R}_j - R_F}{\beta_j} \) greater than \( \phi_\kappa \) from equation (9). Finally the optimal proportion to invest in each security can be found by scaling the \( Z_i \), i.e., \( X_j = Z_i / \sum_{\ell} Z_\ell \).

III. Optimal Portfolios for Constant Coscale Factor Model

In this section we employ the notation of the last section, and assume that \( R \) has a multivariate stable distribution as defined by Press [13, 14]. We further assume the special case of constant coscale factor, i.e., (see [13], [14] for details) if we let \( \Sigma = \Sigma_{i,j} \) denote the scale-coscale matrix, then \( S_{ij} = \rho S_i^2 S_j \) for \( i \neq j \) and \( S_n = S_1^2 \).

More importantly, for a portfolio \( X \),

\[
\bar{R}_X = E(R_X) = \sum_{i=1}^{n} X_i E(R_i) = \sum_{i=1}^{n} X_i \bar{R}_i \tag{11}
\]

and

\[
S_X = \frac{1}{2} (X' \Sigma X)^{\alpha/2} = \frac{1}{2} \left( \sum_{i=1}^{n} X_i^2 S_i^2 + 2 \rho \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_i X_j S_i S_j \right)^{\alpha/2} \tag{12}
\]

Except for the mean and scale parameter of a portfolio \( X \) now given by (11) and (12) respectively, the optimal portfolio is obtained as in Section 2. Thus separation results hold in this case as well and the optimal 'mutual fund' of risky assets is obtained as the optimal solution to the optimization problem (I) with \( \bar{R}_X \) and \( S_X \) now defined by (11) and (12) respectively. Furthermore, analysis in [3] for normal distributions (i.e., \( \alpha = 2 \)) carries over exactly in this case without modifications and will not be repeated here. We recall that the simple sequential procedure described in Section 2 to obtain the optimal portfolio holds in this case as well with \( (\bar{R}_i - R_F)/\beta_i \), in the last section replaced by \( (\bar{R}_i - R_F)\bar{S}_i \).
IV. Conclusions

In this paper, we have shown that the construction of optimal portfolios can be vastly simplified by using simple ranking procedures when returns follow a stable distribution and the dependence structure has any of several standard forms. This ranking procedure simplifies the computations necessary to determine an optimum portfolio. In addition it clarifies the characteristics of the security that make its inclusion in an optimum portfolio desirable. The significance of our results for the nonnormal stable distributions increases when one notes that no other algorithm is known to exist that efficiently obtains the admissible portfolios.

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