Dynamics in Spread Curves

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In this Unit . . .

we will introduce dynamics into the credit spread curves of the previous section. For this we will need some analytical tools, most notably the change-of-measure technique.

But then we will be able to reap the rewards:

- straightforward derivation of the arbitrage-free dynamics of the default hazard rates
- closed-form prices for caps on credit spreads
- closed-form prices for options on CDS
- a better understanding of intricacies like multiple-currency default risk, interest-rate correlation, OTC counterparty risk
Prices and Numeraires

• A price is an *exchange ratio*:
  3 apples for 4 pears, one cow for 5 sheep, … one corporate bond for EUR 108.735.
• An exchange relationship needs a reference point, the *numeraire*. A price set expresses all prices in terms of the *numeraire*.
• The usual numeraire is *cash*.
• Other numeraires:
  * cash tomorrow (\(T\)): \(T\)-forward prices
  * foreign currency prices (USD)
• Prices also represent *choices* and *opportunities*:
  We can choose to either own one corporate bond, or 108.735 USD. In equilibrium, we should be indifferent between these choices.
State Prices and Pricing Probabilities

A simple world with one period and \( N \) states of nature \( \omega_1, \ldots, \omega_N \) tomorrow.

- The state \( \omega_n \) has (subjective) probability \( p_n \).
- A payoff of EUR 1 in state \( \omega_n \) has the value \( \pi_n \).
  These are the state prices.
- Any asset with payoff \( X(\omega_n) \) will be valued as

\[
\sum_{n=1}^{N} \pi_n X(\omega_n).
\]

- In particular, the asset paying \( \omega_n = 1 \) in all states, has value

\[
B = \sum_{n=1}^{N} \pi_n.
\]
There are several components to the *state prices*

\[ \pi_n = p_n L_n B \]

- \( p_n \): a *probability* component
- \( b \): a *discounting* component
- \( L_n \): a *risk premium*.

• Note \( \sum_n p_n L_n = 1 = \mathbb{E}^P [ L ] \) and \( L \geq 0 \). Thus we can define new probabilities

\[ q_n := p_n L_n = \frac{\pi_n}{B}, \quad \mathbb{E}^Q [ X ] = \mathbb{E}^P [ LX ]. \]

• Re-writing the price of the asset with payoff \( X \)
  we have \( x(0)/B(0) = \mathbb{E}^Q [ X(T)/B(T) ] \) because of

\[ \sum_{n=1}^{N} \pi_n X(\omega_n) = \sum_{n=1}^{N} q_n BX(\omega_n) = \mathbb{E}^Q [ BX ] \]
The Theorem of Radon-Nikodym

Let $P$ be one probability measure on $(\Omega, \mathcal{F})$, and $Q \ll P$ another probability measure.

Then there exists a Radon-Nikodym Density $L(\omega)$ with the following properties:

- $L \geq 0$ is nonnegative and integrable: $\mathbb{E}^P[L] = 1$
- All expectations under $Q$ can be represented as

$$
\mathbb{E}^Q[X(\omega)] = \mathbb{E}^P[L(\omega)X(\omega)].
$$

Conversely, every $L \geq 0$ with $\mathbb{E}^P[L] = 1$ defines a probability measure $Q$ on $(\Omega, \mathcal{F})$.

In a dynamic model, $L(t) := \mathbb{E}^P[L | \mathcal{F}_t]$ is called the density process.
The Spot Martingale Measure $Q$

The spot martingale measure $Q$ is the probability measure, under which the discounted security price processes become martingales. The numeraire to the spot-martingale measure is the continuously compounded savings account $b(t)$. Its inverse is the continuously compounded discount factor $\beta(t)$

$$
\beta(t) = e^{-\int_0^t r(s)ds} \quad b(t) = e^{\int_0^t r(s)ds}.
$$

Under the spot-martingale measure $Q$, the time-$t$ price of a random payoff $X$ at time $T_k$ is

$$
p(t) = E^Q \left[ \frac{\beta(T_k)}{\beta(t)} X \bigg| \mathcal{F}_t \right] = E^Q \left[ \frac{b(t)}{b(T_k)} X \bigg| \mathcal{F}_t \right].
$$

Thus $\beta(t)p(t)$, i.e. the price $p(t)$ normalized with the $Q$-numeraire $b(t)$, is a $Q$-martingale.
The Change of Measure / Change of Numeraire Technique

Choose a different asset (with \( Q \)-price \( A(t) \)) as numeraire. Then

\[
p'(t) = p(t) \frac{b(t)}{A(t)}
\]

is the price using the new numeraire.

Using equation (1) the price under the new numeraire is

\[
p'(t) = \frac{b(t)}{A(t)}p(t) = \frac{b(t)}{A(t)} E_Q \left[ \frac{A(T_k)}{b(T_k)} X' \mid \mathcal{F}_t \right]
\]

\[
= E_Q \left[ \frac{L_A(T_k)}{L_A(t)} X' \mid \mathcal{F}_t \right] = E_{P^A} [ X' \mid \mathcal{F}_t ] \tag{2}
\]

where \( X' = X/A(T_k) \) is the payoff (final value) \( X \) of the contingent claim in terms of the new numeraire asset \( A \).
In equation (2) a new pricing measure $P_{A}$ is defined by the Radon-Nikodym density process $L_{A}(t)$

$$\frac{dP_{A}}{dQ}\bigg|_{\mathcal{F}_{t}} = L_{A}(t) := \frac{1}{A(0)} \frac{A(t)}{b(t)}.$$  \hspace{1cm} (3)

Because $A(t)$ is the $b$-price of a traded asset, the process $L_{A}(t)$ is a nonnegative $Q$-martingale with initial value $L_{A}(0) = 1$. $L_{A}(t)$ is therefore a valid Radon-Nikodym density process and $P_{A}$ is a well-defined probability measure.

By equation (2), prices $p'$ in the numeraire $A$ are $P_{A}$-martingales. Thus the calculation of the initial price $p'$ can be reduced to the calculation of the expected final value $X'$ under a changed probability measure $P_{A}$.

$P_{A}$ can be considered to be a set of state prices:
A state security $p_{E}$ for state $E \in \mathcal{F}_{T_{k}}$. By equation (2), $P_{A}[E]$ is the $A$-price $p'_{E}$ of a payoff of 1 units of $A(0)$ in event $E$. 
Libor Market Models

- Miltersen / Sandmann / Sondermann (JoF 1997)
- Brace / Gatarek / Musiela “BGM” (Math. Fin. 1997)
- Jamshidian (Finance and Stochastics 1997)

large and growing follow-up literature.

Reasons for popularity:

❖ positive interest rates

❖ automatic fitting to initial term structure of interest rates

❖ Easy calibration:
  Black (1976) formula for caplets
  excellent approximation (Black (1976)) to swaption prices
Disadvantage:
Path-dependent, relies on Monte-Carlo simulation for pricing.
American and Bermudan options difficult to price.
Model Setup

The Default Model

- $\tau$: time of default
  - e.g. triggered by first jump of Cox process $N(t)$
- $\lambda(t)$: default intensity (can be stochastic)
- $I(t) = 1_{\{\tau > t\}}$: survival indicator function

Tenor Structure

$T_0, T_1, \ldots, T_K$  \hspace{1cm} $\delta_k = T_{k+1} - T_k$

$\kappa(t)$: The next date in the tenor structure after $t$.

\[ T_{\kappa(t)-1} \leq t < T_{\kappa(t)} \]
Bond Prices

as before:

- Default-free zero coupon bonds
  \[ B(t, T_k) = B_k(t) \]
- Defaulatable zero coupon bonds (zero recovery)
  \[ \overline{B}(t, T_k) = \overline{B}_k(t) \]
  Defaultable bond price: \( I(t)\overline{B}_k(t) \)
- Default-risk factors
  \[ D(t, T_k) = D_k(t) = \frac{\overline{B}_k(t)}{B_k(t)}. \]

We do not have independence of interest-rates and defaults any more:

\[ D(t, T) \neq P(t, T) \]

Survival probabilities need not equal ratio of defaulatable to default-free ZCB.
Forward Rates

\[ F_k = \frac{1}{\delta_k} \left( \frac{B_k}{B_{k+1}} - 1 \right), \]
\[ \bar{F}_k = \frac{1}{\delta_k} \left( \frac{\bar{B}_k}{\bar{B}_{k+1}} - 1 \right), \]
\[ H_k = \frac{1}{\delta_k} \left( \frac{D_k}{D_{k+1}} - 1 \right). \]

\[ B_{k+1} = \frac{1}{1 + \delta_k F_k} B_k \]
\[ \bar{B}_{k+1} = \frac{1}{1 + \delta_k \bar{F}_k} \bar{B}_k \]

Note: \( \bar{F}_k = F_k + H_k(1 + \delta_k F_k) \): It is sufficient to specify \( F_k \) and \( H_k \).
Dynamics

All dynamics are driven by one $d$-dimensional BM $W$.

Lognormal diffusion for default-free forward rates ($\sigma^F_k = \text{const}$):

\[
\frac{dF_k}{F_k} = \mu^F_k \, dt + \sigma^F_k \, dW
\]

Lognormal diffusion for discrete hazard rates:

\[
\frac{dH_k}{H_k} = \mu^H_k \, dt + \sigma^H_k \, dW \quad \text{(4)}
\]

with $\sigma^H_k$ constant.
Relations between volatilities:

\[
\sigma_k^H = \sigma_k^S - \frac{\delta_k F_k \sigma_k^F}{1 + \delta_k F_k}
\]

\[
\overline{F}_k \sigma_k^F = \sigma_k^F F_k + \sigma_k^S S_k \\
= (1 + \delta_k F_k) H_k \sigma_k^H + (1 + \delta_k H_k) F_k \sigma_k^F.
\]
**Drift Restrictions Continuous Tenor**

\[ f(t, T) = -\frac{\partial}{\partial T} \ln B(t, T) \]

\[ \bar{f}(t, T) = -\frac{\partial}{\partial T} \ln \bar{B}(t, T) \]

Default-free forward rates: (Heath-Jarrow-Morton 1992):

\[ df(t, T) = \sigma^f(t, T) \left( \int_t^T \sigma^f(t, s)ds \right) dt + \sigma^f(t, T)dW_Q, \]

Defaultable forward rates: (Schönbucher 1998)

\[ d\bar{f}(t, T) = \sigma^{\bar{f}}(t, T) \left( \int_t^T \sigma^{\bar{f}}(t, s)ds \right) dt + \sigma^{\bar{f}}(t, T)dW_Q, \]
\[ \bar{f}(t, t) = \lambda(t) + f(t, t). \]

Note:

- All dynamics under the spot martingale measure \( Q \)
- Every interest-rate model can be represented as a HJM model.
- The driving factor are always the bond price \emph{volatilities}.
- Given the volatilities, the bond prices can be expressed as stochastic exponentials.
Bond Price Dynamics

The bond prices have the following dynamics

\[
\frac{dB_k(t)}{B_k(t)} = r(t)dt - \left( \int_t^{T_k} \sigma_f(t, s) ds \right) dW(t)
\]

\[
\frac{d\overline{B}_k(t)}{B_k(t)} = (r(t) + \lambda(t))dt - \left( \int_t^{T_k} \sigma_{\overline{f}}(t, s) ds \right) dW(t)
\]

\[
\frac{d[\overline{B}_k(t)I(t)]}{B_k(t)} = I(t)(r(t) + \lambda(t))dt - I(t) \left( \int_t^{T_k} \sigma_{\overline{f}}(t, s) ds \right) dW(t) - dI(t).
\]

If the bond price is multiplied with the discount factor \( \beta(t) \), then the drift is reduced by \( r(t)dt \).
Stochastic Exponentials

Given a nonnegative stochastic process $X(t)$ satisfying

$$\frac{dX(t)}{X(t-)} = \mu_X(t)dt + \sigma_X(t)dW(t) - q(t)dN(t),$$

where $X(0) = 0$, $q \leq 1$, regularity on $\mu_X, \sigma_X, N(t)$ a point process with only a finite number of jumps a.s..

Then $X(t)$ is given by

$$X(t) = X(0) \exp \left\{ \int_0^t \mu_X(s) - \frac{1}{2} \sigma_X^2(s)dt + \int_0^t \sigma_X(s)dW(s) \right\} \cdot \prod_{\tau_i \leq t, \ dN(\tau_i) = 1} (1 - q(\tau_i)).$$
The Girsanov Theorem

- $W_Q(t)$: a $n$-dimensional $Q$-Brownian motion
- $N(t)$: a point process with a finite number of jumps
- $\lambda_Q(t)$: the $Q$-intensity of $N(t)$.

Measure Change processes (regularity required):
- $\theta$: a $n$-dimensional predictable process
- $\phi(t)$: a nonnegative predictable process

Define the process $L$ by $L(0) = 1$ and

$$\frac{dL(t)}{L(t-)} = \theta(t)dW_Q(t) + (\phi(t) - 1)(dN(t) - \lambda_Q(t)dt).$$
Then for the probability measure $P$ with

$$dP(t) = L(t)dQ(t)$$

it holds that

$$dW_Q(t) - \theta(t)dt = dW_P(t)$$

defines $W_P$ as $P$-Brownian motion and

$$\lambda_P(t) = \phi(t)\lambda_Q(t)$$

is the intensity of $N$ under $P$. 
Basic Measures

The Subjective Measure $P$

does not take risk premia into account:
Need to change to a pricing measure.

The Spot Martingale Measure $Q$

Discount factor $\beta(t) = e^{-\int_0^t r(s) ds}$.

Under the spot-martingale measure $Q$, the value of a random payoff $X$ at time $T_k$ as seen from $t \leq T_k$ is

$$p(t) = \mathbb{E}^Q \left[ \frac{\beta(T_k)}{\beta(t)} X \middle| \mathcal{F}_t \right].$$

Thus $\beta(t)p(t)$, i.e. the price $p(t)$ normalized with the $Q$-numeraire $1/\beta(t)$, is a $Q$-martingale.
The $T_k$ Forward Measure $P_k$

$$p = B_k(0) \mathcal{E}^Q \left[ \frac{\beta(T_k) B_k(T_k)}{B_k(0)} X \right],$$

Define the Radon-Nikodym density process

$$L_k(t) := \frac{\beta(t) B_k(t)}{B_k(0)} = dP_k \bigg|_{\mathcal{F}_t} dQ$$

for a change of measure from $Q$ to a new measure $P_k$.

$$p = B_k(0) \mathcal{E}^Q \left[ X L_k(T_k) \right] = B_k(0) \int_{\Omega} X L_k(T_k) dQ$$

$$= B_k(0) \int_{\Omega} X \frac{dP_k}{dQ} dQ = B_k(0) \mathcal{E}^{P_k} \left[ X \right].$$
Default Probabilities under $P_k$

For every traded asset, the $T_k$-forward price is a martingale under $P_k$

$$\frac{p(0)}{B_k(0)} = \mathbb{E}^{P_k}[X].$$

In particular: Choose $X = I(T_k)$. Then $p(0) = I(0)\overline{B}_k(0)$.

$$I(0)\frac{\overline{B}_k(0)}{B_k(0)} = I(0)D_k(0)$$

$$= \mathbb{E}^{P_k}[I(T_k)] = P_k[\tau > T_k].$$

- $D_k$ is the $P_k$-survival probability until $T_k$
- $I(t)D_k(t)$ is a $P_k$-martingale.
By Girsanov:

\[ dW_k(t) := dW_Q(t) + \alpha_k(t)dt, \]

\( \alpha_k(t) \) is minus the vector of the volatilities of the default-free zero-coupon bond \( B_k(t) \):

\[ \alpha_k(t) = \int_t^{T_k} \sigma_f(t, s)ds. \]

The default intensity is \textit{not} affected by the change of measure, \( \lambda_Q = \lambda_{P_k} \).

Recurrence Relation between the \( \alpha_k \):

\[ \alpha_{k+1}(t) = \alpha_k(t) + \frac{\delta_k F_k(t)}{1 + \delta_k F_k(t)} \sigma^F_k(t) \]

Change from \( P_k \) to \( P_{k+1} \):

\[ E^{P_k}[X] = \frac{1}{1 + \delta_k F_k(0)} E^{P_{k+1}}[(1 + \delta_k F_k(T_k))X]. \]
The Survival Measure $\bar{P}_k$

The value of a *survival contingent* payoff of $X$ at $T_k$:

$$ p = E^Q \left[ \beta(T_k) I(T_k) X \right] $$

$$ = \bar{B}_k(0) E^Q \left[ \frac{\beta(T_k) I(T_k) \bar{B}_k(T_k)}{\bar{B}_k(0)} X \right], $$

Define the Radon-Nikodym density process for a new measure $\bar{P}_k$:

$$ \bar{L}_k(t) := \frac{\beta(t) I(t) \bar{B}_k(t)}{\bar{B}_k(0)} =: \frac{d\bar{P}_k}{dQ} \bigg|_{\mathcal{F}_t}, $$

$\bar{L}_k$ jumps to zero at default ( $I(\tau) = 0$).

The measure $\bar{P}_k$ attaches a weight of zero to all events that involve default before $T_k$:

$$ \bar{P}_k(\tau \leq T_k) = E^Q \left[ \bar{L}_k(T_k) 1_{\{\tau \leq T_k\}} \right] = 0, $$
The survival measure $\overline{P}_k$ is not equivalent to $Q$ any more, but it is absolutely continuous w.r.t. $Q$, which is all we need to apply Girsanov’s theorem.
Conditional Survival

$\overline{P}_k$ is the $T_k$-forward measure $P_k$, \textit{conditioned on survival} until $T_k$.

Consider an event $A \in \mathcal{F}_{T_k}$:

$$
\mathbb{E}^{P_k} \left[ A \mid \tau > T_k \right] = \frac{\mathbb{E}^{P_k} \left[ AI(T_k) \right]}{\mathbb{E}^{P_k} \left[ I(T_k) \right]} = \frac{\mathbb{E}^{P_k} \left[ AI(T_k) \right]}{D_k(0)}
$$

$$
= \mathbb{E}^{Q} \left[ AI(T_k) \frac{1}{D_k(0)} \frac{\beta(T_k)}{B_k(0)} \right]
$$

$$
= \mathbb{E}^{Q} \left[ A \frac{\beta(T_k) I(T_k)}{B_k(0)} \right]
$$

$$
= \mathbb{E}^{\overline{P}_k} \left[ A \right].
$$

This relationship can provide a basis of a simulation-implementation.
Change of Drift

Girsanov:
The intensity factor is zero $\phi(t) = 0$.

The change of drift factors:

$$d\bar{W}_k(t) := dW_Q(t) + \bar{\alpha}_k(t)dt,$$

where now $\bar{\alpha}_k(t)$ is minus the volatility vector of $\bar{B}_k(t)$

$$\bar{\alpha}_k(t) = \int_t^{T_k} \bar{\sigma}(t, s)ds.$$

Again, the $\bar{\alpha}_k(t)$ are recursively related through

$$\bar{\alpha}_{k+1}(t) = \bar{\alpha}_k(t) + \frac{\delta_k \bar{F}_k(t) \sigma_k(t)}{1 + \delta_k \bar{F}_k(t)}$$
In terms of $H$:

$$\bar{\alpha}_k(t) = \alpha_k(t) + \alpha^D_k(t),$$

where $\alpha^D_k(t)$ is the volatility vector of $D_k(t)$.

The $\alpha^D_k(t)$ are recursively related through

$$\alpha^D_{k+1}(t) = \alpha^D_k(t) + \frac{\delta_k H_k(t) \sigma^H_k(t)}{1 + \delta_k H_k(t)}$$

The change from $\bar{P}_k$ to $\bar{P}_{k+1}$ is done via:

$$\mathbb{E}^{\bar{P}_k}[X] = \frac{1}{1 + \delta_k F_k(0)} \mathbb{E}^{\bar{P}_{k+1}} \left[ (1 + \delta_k F_k(T_k))X \right].$$
Change between $\overline{P}_k$ and $P_k$

$$E^{\overline{P}_k}[X] = \frac{1}{D_k(0)} E^{P_k}[I(t)D_k(t)X]$$

$$= \frac{B_k(0)}{B_k(0)} E^{P_k}\left[I(t)\frac{\overline{B}_k(t)}{B_k(t)}X\right].$$

The Radon-Nikodym density for a change of measure from $P_k$ to $\overline{P}_k$ is thus

$$\frac{d\overline{P}_k}{dP_k} = \frac{B_k(0)}{\overline{B}_k(0)}I(t)\frac{\overline{B}_k(t)}{B_k(t)} = \frac{D_k(t)}{D_k(0)}I(t).$$

The relation between the Brownian motions under the $T_k$ forward measure and the
$T_k$ survival measure is (for $t < T_k$)

$$d\overline{W}_k(t) = dW_k(t) + \alpha_k^D(t)dt.$$
Changing back from $\overline{P}_k$ to $P_k$

If the Cox process properties of $N(t)$ are used and $X$ does not contain any direct reference to defaults, it is possible to change measure from a variant of $\overline{P}_k$ back to $P_k$:

$$\mathbf{E}^{\overline{P}_k}[X] = \frac{B_k(0)}{B_k(0)} \mathbf{E}^{P_k} \left[ e^{-\int_0^t \lambda(s) ds} \frac{B_k(t)}{B_k(t)} X \right],$$

$$\mathbf{E}^{\overline{P}_k}[X] = \mathbf{E}^{P_k} \left[ \mathcal{E} \left( \int_0^{T_k} \alpha_k^D(s) dW_k(s) \right) X \right] =: \mathbf{E}^{P_k} \left[ L_k^D(T_k) X \right].$$
Radon-Nikodym density:

\[
\frac{d\bar{P}'_k(t)}{dP_k(t)} = L^D_k(t) = e^{-\int_0^t \lambda(s) ds} \frac{B_k(t)B_k(t)}{B_k(t) \bar{B}_k(0)}
\]

\[
= \frac{\gamma(t)D_k(t)}{D_k(0)},
\]

where \( \gamma(t) = e^{-\int_0^t \lambda(s) ds} \). In particular, \(1/(\gamma(t)D_k(t))\) is a \(\bar{P}'_k\)-martingale and therefore it is also a \(\bar{P}_k\)-martingale.
Independence

Definition:

*Independence* . . . between the default-free bond prices $B_k$ and defaults means:

- Independence (under $Q$) of $B_k(t)$ and $I(t')$
- Independence (under $Q$) of $B_k(t)$ and $\mathbb{E}^Q[ I(t') ]$
- $\mathbb{E}^Q[ I(T_l) ] = D_l(0)$
- Independence (under $Q$) of $B_k(t)$ and $D_l(t')$
- Zero covariation between $F$ and $H$

$$\alpha_k(t)\alpha_l^D(t) = 0 \quad \forall k, l$$
Drift Restrictions

Default-Free Forward Rates:

$B_k / B_{k+1}$ is a martingale under the $T_{k+1}$-forward measure.

$$F_k = \frac{1}{\delta_k} \left( \frac{B_k}{B_{k+1}} - 1 \right)$$

is a martingale under the $T_{k+1}$-forward measure.

$$dF_k(t) = F_k(t) \sigma^F_k \, dW_{k+1}(t).$$
Defaultable Forward Rates:

\[ \frac{B_k}{B_{k+1}} \] is a martingale under the \( T_k \)-survival measure.

\[ F_k = \frac{1}{\delta_k} \left( \frac{B_k}{B_{k+1}} - 1 \right) \]

is a martingale under the \( T_{k+1} \)-survival measure.

\[ dF_k(t) = F_k(t) \sigma_k dW_{k+1}(t). \]
Forward Spreads:

The dynamics of the forward spreads under the $T_{k+1}$ survival measure are

$$dS_k = F_k \sigma_k^F \alpha_k^{D_{k+1}} dt + S_k \sigma_k^S d\overline{W}_{k+1}.$$ 

Forward Intensities:

Dynamics under $\overline{P}_{k+1}$:

$$dH_k = F_k \sigma_k^F \left( \frac{1 + \delta_k H_k}{1 + \delta_k F_k} \alpha_k^{D_{k+1}} - \delta_k H_k \sigma_k^H \right) dt$$

$$+ H_k \sigma_k^H d\overline{W}_{k+1}.$$
Independence:

Under independence $\sigma^F \alpha^D = 0 = \sigma^S \sigma^F$:

$$
\begin{align*}
    dF_k &= F_k \sigma^F_k d\bar{W}_{k+1} \\
    d\bar{F}_k &= \bar{F}_k \sigma^\bar{F}_k d\bar{W}_{k+1} \\
    dS_k &= S_k \sigma^S_k d\bar{W}_{k+1} \\
    dH_k &= H_k \sigma^H_k d\bar{W}_{k+1}
\end{align*}
$$

The discrete default intensities $H_k$, the credit spreads $S_k$, the defaultable forward rates $\bar{F}_k$ and the default-free forward rates $F_k$, in short all forward rates with fixing at $T_k$ are martingales under the $T_{k+1}$-survival measure.

If $H_k$ and $F_k$ are independent under $Q$, then this independence is preserved and $S_k$ and $F_k$ are also independent under $P_{k+1}$.
Recovery Payoffs under Correlation

Define

\[ A_k^D := - \sum_{l=0}^{k-1} \frac{\delta_l H_l \sigma_l^H}{1 + \delta_l H_l} (T_l - T_0). \]

(i) **Fixed Payment at Default:**
The value of a payment of 1 at \( T_{k+1} \) if a default happens in \( [T_k, T_{k+1}] \) is approximatively

\[
e_k := \delta_k H_k \overline{B}_{k+1}
\]

\[
+ \overline{B}_k \text{cov}^P_k \left( \frac{1}{1 + \delta_k F_k(T_k)}, \frac{\delta_k F_k}{1 + \delta_k F_k} \right)
\]

\[
\approx \delta_k H_k \overline{B}_{k+1} - \overline{B}_k \frac{\delta_k F_k}{1 + \delta_k F_k} \left( \exp \left\{ \frac{\sigma_k^F A_k^D}{1 - \delta_k F_k} \right\} - 1 \right),
\]
(ii) **Floating Payment at Default:**

The value of \( 1 + \delta_k F_k(T_k) \) at \( T_{k+1} \) if a default occurs in \( [T_k, T_{k+1}] \) is approximatively

\[
\approx \delta_k \overline{B}_{k+1} \left[ \overline{F}_k - F_k \exp \left\{ A^D_{k+1,k} \sigma^F_k \right\} \right].
\]

(iii) **Floating Coupon in Survival:**

The value of \( F_k(T_k) \) at \( T_{k+1} \) if no default occurs until \( T_{k+1} \) is approximatively

\[
\approx \overline{B}_{k+1} F_k \exp \left\{ A^D_{k+1} \sigma^F_k \right\}
\]

With these results we can value

- defaultable fixed coupon bonds
- defaultable floating coupon bonds
- default swaps
Default Swap Valuation

Specification

- **A** pays $\overline{s}$ at $T_i$ until $T_N$ or default (fee stream)
- **B** pays the difference between the post-default price of the reference asset (a bond issued by **C**) and its par value at default (default payment).

**The Fee**

The value of the fee stream can be directly determined as

$$\overline{s} \sum_{k=1}^{N} \delta_{k-1} \overline{B}_k(0)$$

This is valid for all fee streams of credit derivatives that pay fees until default.

(Note: At a default, the accrued fee is usually paid at default. The fee should be multiplied with daycount fractions.)
The Default Payment

The value of receiving 1 at default is

\[ \sum_{k=0}^{N-1} e_k. \]

The value of receiving \(1 - \pi\)

\[ (1 - \pi) \sum_{k=0}^{N-1} e_k. \]
The Default Swap Rate

\[ \bar{s} = (1 - \pi(1 + c)) \sum_{k=0}^{N-1} \overline{w}_k \delta_k H_k \]

(for independence), and

\[ \bar{s} \approx (1 - \pi(1 + c)) \sum_{k=0}^{N-1} \overline{w}_k \delta_k \left[ H_k - F_k \frac{1 + \delta_k H_k}{1 + \delta_k F_k} \left( \exp \left\{ \frac{\sigma_k^{FD} A_k^D}{1 - \delta_k F_k} \right\} - 1 \right) \right], \]

under correlation, where \( \overline{w}_k := \overline{B}_{k+1} / \sum_{j=0}^{N-1} \overline{B}_{j+1} \)

Compare to this:
The fixed-for-floating interest rate swap rate:

\[ s = \sum_{k=0}^{N-1} w_k \delta_k F_k \]

with weights \( w_k := B_{k+1} / \sum_{j=0}^{N-1} B_{j+1} \).
Marking-to-Market of an existing CDS

- Existing CDS, CDS-rate: $s'$
- Current market rate for identical CDS: $\bar{s}$

Entering an offsetting position yields the fee difference until default.
The value of a long position in the CDS with $s'$ is

$$(\bar{s} - s') \sum_{k=0}^{N-1} B_{k+1}.$$ 

(Short position: negative.)
Options on Default Swaps

Dynamics of the *forward* Default Swap Rate:

\[ d\bar{s} = \ldots dt + \sigma \bar{s}dW \]

Using the numeraire

\[ X(t) := I(t) \sum_{k=K}^{N-1} \delta_{k-1} B_k(t) \]

\( \bar{s}(t)X(t) \) becomes a traded asset, and thus \( \bar{s} \) is a martingale under the associated measure, the *default swap measure* \( \bar{P}^s \):

\[ d\bar{s} = \sigma \bar{s} dW^s \]

The *default swap measure* is a survival contingent measure that attaches weight
zero to defaults before $T_K$. Its density is given by

$$\frac{d\bar{P}_{k}^{s}}{dQ}(t) = L^{s}(t) = \frac{\beta(t) X(t)}{\beta(0) X(0)}.$$
A Black-Type Formula for Options on Default Swaps

Option to enter at time $T_K$ a default swap over $[T_K, T_N]$ at a strike spread of $\bar{s}^*$
(Option is knocked out at default before $T_K$)

$$C = \mathbb{E}^Q \left[ \beta(T_K) I(T_k) \sum_{k=K}^{N-1} \delta_{k-1} B_k(T_K) (\bar{s}(T_K) - \bar{s}^*)^+ \right]$$

the change of measure yields

$$C = X(0) \mathbb{E}^{\overline{P}^s} \left[ (\bar{s}(T_K) - \bar{s}^*)^+ \right].$$

$d\bar{s} = \bar{s}\sigma dW^s$ is lognormally distributed under $\overline{P}^s$:

$$C = X(0) \left\{ \bar{s}(0) N(d_1) - \bar{s}^* N(d_2) \right\}$$
where

\[ d_{1;2} = \frac{\ln(\bar{s}/\bar{s}^*) \pm \sigma^2 T_K}{\sigma \sqrt{T_K}}. \]

Note: This formula holds *irrespective of recovery rates / correlations*.

Using the so-called swaption approximation, the formula can also be derived in the Libor-based setup.
Numerical Implementation

- $C(T_k)$ be the value of a credit derivative given $\tau > T_k$
- $X_k$ its payoff at $T_{k+1}$ if a default occurs in $[T_k, T_{k+1}]$

\[
C(T_k) = \mathbf{E}^Q \left[ \frac{\beta(T_{k+1})}{\beta(T_k)} C(T_{k+1}) \right]
\]

\[
= B_{k+1}(T_k) \mathbf{E}^{P_{k+1}} \left[ C(T_{k+1})I(T_{k+1}) \right]
+ B_{k+1}(T_k) \mathbf{E}^{P_{k+1}} \left[ C(T_{k+1})(1 - I(T_{k+1})) \right]
\]

\[
= \ldots
\]

\[
= \overline{B}_{k+1}(T_k) \mathbf{E}^{\overline{P}_{k+1}} \left[ C(T_{k+1}) \right]
+ B_{k+1}(T_k)(1 - D_{k+1}(T_k))X_{k+1}.
\]
Default Tree Simulation

Starting from $T_k$ the simulation until $T_{k+1}$ proceeds as follows:

- $P_{k+1}$ survival probability until $T_{k+1}$ is $D_{k+1}(T_k)$. This is the probability on the survival branch.

- $1 - D_{k+1}(T_k)$ probability on the default branch.

- Add the value in default weighted with default probability.

- Add any survival payoffs (with survival probability).

- Simulate $F, H$ until $T_{k+1}$ on the survival branch of the tree under the $T_{k+1}$ survival measure $P_{k+1}$.

The simulation takes place under the survival spot Libor measure.
Bermudan Options on Default Swaps

exercise dates: $T_K, T_{K+1}, \ldots, T_N$

- Bermudan on CDS: Option to enter a CDS at one of the dates
- alternative: CDS with option to cancel/extend at each date
- possibly step-up in the fee
- unwound at default

useful for
- hedging of credit line commitments
- management of regulatory capital.

Only numerical solution possible.
Preparation for the $T_N$-Survival Measure

1. Convert Payoffs in default to payoffs in survival:
   $X$ at $T_{k+1}$ iff default in $[T_k, T_{k+1}]$ becomes

   $$B_{k+1}(T_k)\mathbb{E}^{P_{k+1}}\left[ X \mid \mathcal{F}_{T_k} \right] - \overline{B}_{k+1}(T_k)\mathbb{E}^{\overline{P}_{k+1}}\left[ X \mid \mathcal{F}_{T_k} \right]$$

   the equivalent pre-default value iff survival until $T_k$.

2. Convert payoffs in survival into $\overline{B}_N$-units.
   (Hypothesis: Re-investment in $\overline{B}_N$)

   Don’t worry, usually $X$ is very simple: constant or known at $T_k$. 
Preparation of the Bermudan CDS-Option

Option to . . .

- *enter* a CDS as protection buyer
- over $[T_k, T_N]$ (from exercise date until $T_N$)
- at a fixed rate $\bar{s}^*$
- at each date $T_k$ in $T_0, \ldots, T_N$. 
Payoff of Bermudan CDS-Option

Payoff if exercised at $T_k$

$= \text{value of CDS at } T_k$

$= \left[ \text{value of payoff in default} \right] - \left[ \text{value of strike fee} \right]$

$= \left[ \text{value of fee in market} \right] - \left[ \text{value of strike fee} \right]$

The last term only contains survival contingent values:

$$\text{exercise payoff} = (\bar{s}(T_k) - \bar{s}^*) \sum_{n=k+1}^{N} \overline{B}_n(T_k)$$

The numerical solution can now follow algorithms for Bermudan interest-rate swaptions that have been proposed in the literature. (See e.g. Longstaff/Schwartz (2000), Glasserman/Zhao (1997), Andersen (1999), Broadie/Glasserman (1997), Pedersen (1999))
Conclusion

We showed:

✔ how to adapt the market-model framework to incorporate default risk
  ◦ Libor-based market models
  ◦ Swap-based market models

✔ the appropriate measures: survival contingent forward and swap measures

✔ the numerical implementation strategy

Open problems are mainly in the approximation formulae for correlation between default risk and interest rate risk.