Options and Derivatives

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Overview

- Option basics and option strategies
- No-arbitrage bounds on option prices
- Binomial option pricing
- Black-Scholes-Merton Formula

Option Basics

- Derivatives
- Option characteristics
- Value of options at expiration
- Option strategies
Derivatives

- A derivative is a security with a payoff that depends on the price of another security
- The other security is called the underlying (security)
- Examples: options, futures, swaps
- Derivatives are used for
  - hedging
  - risk management
  - portfolio insurance
  - fine tuning a portfolio
  - speculation

Options Characteristics

- Option types
  - call options
  - put options
- Exercise price or strike price
- Expiration
- European vs. American option
- In-the-money, out-of-the-money, at-the-money
- Price or premium
- Net profit

Value of Options at Expiration

- At expiration, if the stock price is $S_T$, a call option with strike price $X$ is worth:
  \[ C_T = \begin{cases} 
  S_T - X & \text{if } S_T > X \\
  0 & \text{if } S_T \leq X 
  \end{cases} \]
- At expiration, if the stock price is $S_T$, a put option with strike price $X$ is worth:
  \[ P_T = \begin{cases} 
  0 & \text{if } S_T \geq X \\
  X - S_T & \text{if } S_T < X 
  \end{cases} \]
Option Strategies

- Using calls for leverage
- Protective put
- Covered calls
- Straddle
- Collars

Call Options for Leverage

- Example:
  - Microsoft share price is $S_0=80$
  - A call option with $X=$80 and 6-month maturity costs $C_0=$10
  - Risk free rate is 0%
- You have $8,000 and consider 3 strategies
  A. Buy 100 shares of Microsoft
  B. Buy 800 shares, financed by borrowing $56,000
  C. Buy 800 call options
  D. Buy 100 call options and invest $7000 at the risk-free rate

Protective Put

- You buy a share in Microsoft for $S_0=80$
- You are afraid that the stock price will drop.
- How do you limit your possible losses by trading options?
- Make diagrams for values at expiration and profits.
Covered Call

- Suppose that you buy a share of XYZ for \( S_0 = $50 \)
- You think that out-of-the-money call options seem excessively expensive, and you want to profit from this.
- How do you do this?
- What is the value and profit at expiration?
- Compare: “naked option writing”

Straddle

- You have private information that a particular stock’s price will change dramatically soon, but you do not know if it will go up or down.
- Give an example of when this could happen.
- Which option strategy could you use to profit from this information?

Collar

- Suppose that
  - you want to buy 10,000 shares of stock XYZ for $80 per share
  - you want to make sure that, in 6 months, the value of your position is at least $70 per share
  - You think that the stock price will go up, but that it will at most reach (about) $90
- Which option position would be helpful in this situation?
No-Arbitrage Bounds on Option Prices

- Intrinsic value and time value
- Adjusted intrinsic value
- Lower bounds on option prices
- Put-Call Parity

Intrinsic Value of a Call Option

- The intrinsic value is:
  - the value of the right to exercise now
- What is the intrinsic value of an out-of-the-money call?
- What is the intrinsic value of an in-of-the-money call?
- Which is greater:
  - the intrinsic value or the option price?
  - The difference between the option price and the intrinsic value is called the time value of the option.

Adjusted Intrinsic Value

- The adjusted intrinsic value is the present value of the strategy to exercise at expiration for sure.
  \[
  \max(0, S_0 - X/(1+R_f)^T)
  \]
- Assume that the stock does not pay dividends before the expiration of the option, i.e. time T
- What is greater:
  - the option price or the adjusted intrinsic value?
- Conclusions:
  1. Never exercise a call option before T! (This may change if there are dividend payments.)
  2. A lower bound on a call option value is:
     \[
     C_0 \geq \max(0, S_0 - X/(1+R_f)^T)
     \]
Put-Call Parity: Stocks with No Dividends Before Expiration

- Can you create a "synthetic" stock by buying a portfolio of European options and risk-free assets?
- What is the price of such a portfolio?
- Pricing by arbitrage gives the famous put-call parity for European options with strike price $X$, and expiration at time $T$:

$$ C_T - P_T = S_0 - \frac{X}{(1 + R_f)^T} $$

Put-Call Parity: Stocks with Dividends

- The general put-call parity for European options is:

$$ C_T - P_T = S_0 - \frac{X}{(1 + R_f)^T} - PV_d(\text{dividends before time } T) $$

- Note that this gives a relation between put and call prices, which much hold at all times.
- Hence, if we can price a call, we automatically know the price of a put.

Binomial Option Pricing

- Pricing of call options on non-dividend paying stocks
- Methodology works, however, for all derivatives !!!
- Two-state option pricing
- Replication
- Hedge ratio
- Option pricing in a "tree"
- Dynamic hedging
Binomial Option Pricing:
Two-state option pricing
1. Assume that the stock price at expiration can have 2 possible values (2 scenarios)
2. Compute the option payoff in each scenario
3. Replicate the option payoff with a portfolio of stocks and risk-free securities
4. Compute the price of the replicating portfolio
5. This is the option price (because of no arbitrage)!

Hedge Ratio or “Delta”
- The number of stocks in the replicating portfolio (or hedge portfolio) is called the hedge ratio or delta, \( \Delta \)
- The hedge ratio tells you how much the option price changes per unit of change in the stock price:
  \[
  \Delta = \frac{C^* - C^-}{S^* - S^-}
  \]

Binomial Option Pricing:
Option Pricing in a “Tree”
1. Assume that the stock price evolves over time in a “tree”: Every sub-period the stock price can go up or down. (Many scenarios at expiration.)
2. Compute the option payoff at expiration in each scenario
3. Replicate the option payoff with a dynamic hedging strategy using stocks and risk-free securities
4. Compute the initial price of the replicating strategy
5. This is the option price (because of no arbitrage)
Binomial Pricing of Any Derivative

1. Assume that the underlying evolves over time in a “tree”
2. Compute the derivative payoff at expiration in each scenario
3. Replicate the derivative payoff with a dynamic hedging strategy using the underlying and risk-free securities
4. Compute the initial price of the replicating strategy
5. This is the derivative’s price (because of no arbitrage)
   (For American-type derivatives, you should check at every “node” whether exercise is optimal.)

Black-Scholes-Merton Formula

- Assumptions
- “The” formula
- Intuition
- Implied volatility
- Hedge ratio
- Put options

Assumptions

1. The risk-free interest rate is:
   - constant
   - continuously compounded
2. The stock price
   - is log-normal
   - has no jumps
   - has constant volatility
3. The stock pays a constant dividend yield
4. The stock and risk-free security can be traded all the time at no cost
Black-Scholes-Merton Formula

- If we have more and more scenarios in the binomial model, the option price gets more and more precise.
- In the limit, we get the celebrated formula for the price of a European call option:

\[ C_0 = S_0 e^{-rT} N(d_1) - X e^{-rT} N(d_2) \]

\[ d_1 = \frac{\ln(S_0 / X) + (r - \delta + \sigma^2 / 2)T}{\sigma \sqrt{T}} \]

\[ d_2 = d_1 - \sigma \sqrt{T} \]

Intuition

- If the option is almost certain to be in-the-money at maturity, then
  - \( N(d_1) \cong N(d_2) \cong 1 \), and
  - the option price is adjusted intrinsic value, \( S_0 e^{-rT} - X e^{-rT} \)
  - Under which circumstances does this happen?
- If the option is almost certain to be out-of-the-money at maturity, then
  - \( N(d_1) \cong N(d_2) \cong 0 \), and
  - the option price is close to 0
- In general, the price in the risk-adjusted expected payment at maturity

Determinants of Option Values

- Stock price (today), \( S_0 \)
- Exercise price, \( X \)
- Volatility of stock, \( \sigma \)
- Time to expiration, \( T \)
- Interest rate, \( r \)
- Dividend rate of stock, \( \delta \)
**Implied Volatility**

- For every level of volatility, $\sigma$, there is a corresponding option price, $C_0$.
- Similarly, for any option price, $C_0$, there is a corresponding volatility, $\sigma$.
- This is called the "implied volatility".
- According to the Black-Scholes-Merton model, what should be true about the implied volatility of all options on the same stock?

**Hedge Ratio or Delta**

- The number of stocks in the replicating portfolio is called the hedge ratio or the delta.
- It is $\Delta = \frac{\partial C}{\partial S} = e^{-rT} N(d_1)$.
- If an investment bank writes an option to a client, the bank will hedge its position by buying $\Delta$ shares.
- Since $\Delta$ is changing over time, the bank must keep adjusting the number of shares held. This is called dynamic hedging.

**Black-Scholes-Merton Pricing of a Put Option**

- Recall the put-call parity.
- Using the put-call parity, the Black-Scholes-Merton price of a European put option is:

$$ P = C_0 - S_0 + P_F(X) + P_V(d_1) \text{(dividends before time } T) $$

$$ = C_0 - S_0 + X e^{-rT} + S_0 (1 - e^{-rT}) $$

$$ = X e^{-rT} (1 - N(d_1)) - S_0 e^{-rT} (1 - N(d_1)) $$