**Construction of Multi-period Volatility Forecasts using GARCH(1,1)**

**Summary**

There are three main steps to implementation of the GARCH(p,q) model. First, estimate the regression parameters and the parameters of the GARCH process. Second, construct a sequence of the single period forecasts of the variance. This provides us with a set of (daily) volatility forecasts. Third, we need to combine these (daily) into an estimate of volatility for a longer period. This note deals with the second and the third steps of the process, then deals with some advantages and disadvantages of the approach.

**GARCH(1,1) Process**

We are using the following model of the volatility process.

\[ \sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \]

The model states that the variance (volatility squared) at any day is simply equal to a constant, some weight for the squared innovation (yesterday’s error from the regression model) and some additional weight for yesterday’s variance (conditional variance). This is because we are using daily returns. If we were using some other period (for example weekly), we would utilize last week’s squared error and last week’s variance. Also note that this requires that \( \alpha_1 + \beta_1 < 1 \) or else the volatility would expand without bound\(^1\).

This approach captures the persistence of a “shock” to the volatility estimate. This is captured by \( \alpha_1 \) which measures the impact of the shock on current volatility and by \( \beta_1 \) which measures the rate at which the effects dissipate.

To provide a numerical example, suppose that we run some regression and find for our data set that \( \alpha_0 = 0.3, \ \alpha_1 = 0.10, \ \text{and} \ \beta_1 = 0.89 \). Further suppose that we know the current value for the conditional variance. Suppose that this is 0.31. We can then apply the following formula.

\[
E_t[\sigma_{i+j}^2] = (\alpha_1 + \beta_1)^j \left( \sigma_i^2 - \frac{\alpha_0}{1 - \alpha_1 - \beta_1} \right) + \left( \frac{\alpha_0}{1 - \alpha_1 - \beta_1} \right)
\]

Suppose that we want a volatility that is 10 days (or periods) ahead (\( j = 10 \)). Via direct substitution, we can write the following equation:

\[
E_t[\sigma_{i+j}^2] = (0.10 + 0.89)^{10} \left( 0.31 - \frac{0.30}{1 - 0.10 - 0.89} \right) + \left( \frac{0.30}{1 - 0.10 - 0.89} \right)
\]

or after crunching the numbers

\[
E_t[\sigma_{i+j}^2] = (0.904382)(-29.29) + (30) = 3.148898
\]

It should be noted at this point that the unconditional variance is given by:

\[^1\] There is a special case in which \( \alpha_1 + \beta_1 = 1 \). This implies that a shock to volatility persists indefinitely. If this is the situation then we can apply the formula

\[
E_t[\sigma_{i+j}^2] = \sigma_i^2 + j\alpha_0.
\]
\[ \alpha^2 = \left( \frac{\alpha_0}{1 - \alpha_1 - \beta_1} \right) \]

Note that as \( j \to \infty \), the term \((\alpha_1 + \beta_1)^j \to 0\), and the variance converges to the unconditional variance. For a sufficiently long time horizon, we can simply use the unconditional variance.

**Combining single period forecasts**

The above process provides the estimate of the single period variance at any given point in the future. However, we will typically be interested not in the volatility at that single point (day), but rather the volatility over a period of time (for example from now to next week). Clearly we can iterate the above process to obtain a sequence of daily variances. However, we then need to combine them into a volatility estimate for the entire period.

We know that for a Weiner process (Brownian Motion), the variance is \( \sigma^2 \tau \), where \( \tau \) is the time from now to the point of interest, expressed as a fraction of a year (days/365). This can be expressed as:

\[ \sigma^2 \tau = \sum_{i=0}^{n} \sigma_i^2 \tau_i \]

Thus if we have a set to three daily variance forecasts (tomorrow and the next two days), we can combine these into a forecast of the variance over the entire period. Suppose that the forecasts are 0.25, 0.29 and 0.31, respectively. We can then write:

\[ \sigma^2 \left( \frac{3}{365} \right) = 0.25 \left( \frac{1}{365} \right) + 0.29 \left( \frac{1}{365} \right) + 0.31 \left( \frac{1}{365} \right) = 0.85 \left( \frac{1}{365} \right). \]

Or

\[ \sigma^2 = 0.85 \left( \frac{1}{365} \right) / \left( \frac{3}{365} \right) = 0.28333. \]

This result represents the forecast average variance over the period. The volatility is simply the square root of this result.

**Advantages and Disadvantages**

This approach has an advantage over a constant volatility approach as it appears to capture some aspects of the electricity market, specifically the observation that a shock to volatility tends to persist for a period of time. However, this approach does not explicitly allow the introduction of new shocks outside the period we used to fit the model. These need to be applied as exogenous events. This then requires that we estimate the shock process.