Problem 1:

In class, we solved the following deterministic optimal control problem

\[
\min_{\|u\| \leq 1} J(t_0, x_0, u) = \frac{1}{2} (x(\tau))^2
\]

subject to \( \dot{x}(t) = u(t), \quad x(t_0) = x_0 \)

where \( \|u\| = \max_{0 \leq t \leq \tau} \{|u(t)|\} \) using the method of characteristics. In particular, we solved the open-loop HJB PDE equation

\[
W_t(t, x; u) + W_x(t, x; u) u = 0, \quad W(\tau, x; u) = \frac{1}{2} x^2.
\]

for a fixed \( u \) and then find the optimal close-loop control solving

\[
u^*(t, x) = \arg \min_{\|u\| \leq 1} W(t, x; u)\]

and computing the value function as \( V(t, x) = W(t, x; u^*(t, x)) \).

a) Explain why this methodology does not work in general. Provide a counter example.

b) What specific control problems can be solve using this open-loop approach.

c) Propose an algorithm that uses the open-loop solution to approximately solve a general deterministic optimal control problem.

Problem 2: (Dynamic Pricing in Discrete Time)

Assume that we have \( x_0 \) items of a ceratin type that we want to sell over a period of \( N \) days. At each day, we may sell at most one item. At the \( k^{th} \) day, knowing the current number \( x_k \) of remaining unsold items, we can set the selling price \( u_k \) of a unit item to a nonnegative number of our choice; then, the probability \( q_k(u_k) \) of selling an item on the \( k^{th} \) day depends on \( u_k \) as follows:

\[
q_k(u_k) = \alpha \exp(-u_k)
\]

where \( 0 < \alpha < 1 \) is a given scalar. The objective is to find the optimal price setting policy so as to maximize the total expected revenue over \( N \) days. Let \( V_k(x_k) \) be the optimal expected cost from day \( k \) to the end if we have \( x_k \) unsold units.

a) Assuming that for all \( k \), the value function \( V_k(x_k) \) is monotonically nondecreasing as a function of \( x_k \), prove that for \( x_k > 0 \), the optimal prices have the form

\[
\mu_k^*(x_k) = 1 + J_{k+1}(x_k) - V_{k+1}(x_k - 1)
\]
and that

\[ V_k(x_k) = \alpha \exp(-\mu^*_k(x_k)) + V_{k+1}(x_k). \]

b) Prove simultaneously by induction that, for all \( k \), the value function \( V_k(x_k) \) is indeed monotonically nondecreasing as a function of \( x_k \), that the optimal price \( \mu^*_k(x_k) \) is monotonically nonincreasing as a function of \( x_k \), and that \( V_k(x_k) \) is given in closed form by

\[
V_k(x_k) = \begin{cases} 
(N - k) \alpha \exp(-1) & \text{if } x_k \geq N - k, \\
\sum_{i=k}^{N-x_k} \alpha \exp(-\mu^*_i(x_k)) + x_k \alpha \exp(-1) & \text{if } 0 < x_k < N - k, \\
0 & \text{if } x_k = 0.
\end{cases}
\]

**Problem 3:**

Consider a deterministic optimal control problem in which \( u \) is a scalar control and \( x \) is also scalar. The dynamics are given by

\[ f(t, x, u) = a(x) + b(x)u \]

where \( a(x) \) and \( b(x) \) are \( C^2 \) vector functions. If \( P(t) b(x(t)) = 0 \) on a time interval \( \alpha \leq t \leq \beta \), the Hamiltonian does not depend on \( u \) and the problem is *singular*. Show that under these conditions

\[ P(t) q(x) = 0, \quad \alpha \leq t \leq \beta, \]

where \( q(x) = b_x(x)a(x) - a_x(x)b(x) \). Show further that if

\[ P(t)[q_x(x(t))b(x(t)) - b_x(x(t))q(x(t))] \neq 0 \]

then

\[ u(t) = -\frac{P(t)[q_x(x(t))a(x(t)) - a_x(x(t))q(x(t))]}{P(t)[q_x(x(t))b(x(t)) - b_x(x(t))q(x(t))]} . \]

**Problem 4:** (Optimal Learning).

The objective of this note is to characterize a particular family of *Learning Function*. These learning functions are useful modelling devices for situations where there is an agent that tries to increase his or her level of “knowledge” about a certain phenomenon (such as customers’ preferences or product quality) by applying a certain control or “effort”. To fix ideas, in what follows knowledge will be represented by the variable \( x \) while effort will be represented by the variable \( e \). For simplicity we will assume that knowledge takes values in the \([0, 1]\) interval while effort is a nonnegative real
variable. The family of learning function that we are interested in this note are those than can be derived from a specific subfamily that we called Additive Learning Functions. The formal definition of an Additive Learning Function* is as follows.

**Definition 1** Consider a function \( L : \mathbb{R}_+ \times [0,1] \rightarrow [0,1] \). The function \( L \) would be called Additive Learning Function if it satisfies the following properties:

- **Additivity:** \( L(e_2 + e_1, x) = L(e_2, L(e_1, x)) \) for all \( e_1, e_2 \in \mathbb{R}_+ \) and \( x \in [0,1] \).
- **Boundary Condition:** \( L(0, x) = x \) for all \( x \in [0,1] \).
- **Monotonicity:** \( L_e(t, x) = \frac{\partial L}{\partial e} L(e, x) > 0 \) for all \((e, x) \in \mathbb{R} \times [0,1] \).
- **Satiation:** \( \lim_{e \to \infty} L(e, x) = 1 \) for all \( x \in [0,1] \).

a) Prove the following. Suppose that \( L(e, x) \) is a \( C^1 \) additive learning function. Then \( L(e, x) \) satisfies

\[
L_e(e, x) - L_e(0, x) L_x(e, x) = 0
\]

where \( L_e \) and \( L_x \) are the partial derivatives of \( L(e, x) \) with respect to \( e \) and \( x \) respectively.

b) Using the method of characteristics solve the PDE of part a) as a function of

\[
H(x) = \int -\frac{1}{L_e(0, x)} \, dx
\]

and prove that the solution is of the form

\[
L(e, x) := H^{-1}(H(x) - e).
\]

Consider the following optimal control problem.

\[
V(0, x) = \max_{p_t} \int_0^T [p_t \lambda(p_t) x_t] \, dt
\]

subject to \( \dot{x}_t = L_e(0, x_t) \lambda(p_t) \) \( x_0 = x \in [0,1] \) given. (1)

\[
\dot{x}_t = L_e(0, x_t) \lambda(p_t) \quad x_0 = x \in [0,1] \text{ given.} \quad (2)
\]

Where \( L_e(0, x) \) is the partial derivative of the learning function \( L(e, x) \) with respect to \( e \) evaluated at \((0, x)\). This problem corresponds to the case of a seller that tries to maximize cumulative revenue during the period \([0, T] \). Potential demand rate at time \( t \) is given by \( \lambda(p_t) \) where \( p_t \) is the price set by the seller at time \( t \). However, only a fraction \( x_t \in [0,1] \) of the potential customers buy the product at time \( t \). The dynamics of \( x_t \) are given by (2).

c) Show that equation (2) can be rewritten as

\[
x_t = L(y_t, x) \quad \text{where } y_t := \int_0^t \lambda_s \, ds.
\]

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*This name is probably not standard since I do not know the relevant literature well enough.
and use this fact to reformulate your control problem as follows

$$ \max_{y_t} \int_0^T p(\dot{y}_t) \dot{y}_t L(y_t, x) \, dt \quad \text{subject to } y_0 = 0. \tag{3} $$

d) Deduce that the optimality conditions in this case are given by

$$ \dot{y}_t^2 p'(\dot{y}_t) L(y_t, x) = \text{constant}. \tag{4} $$

e) Solve the optimality condition for the case

$$ \lambda(p) = \lambda_0 \exp(-\alpha p) \quad \text{and} \quad L(e, x) = 1 + (x - 1) \exp(-\beta e), \quad \alpha, \beta > 0. $$

Problem 5:

a) Let $M_t$ be a $\mathcal{F}_t$ martingale and let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function such that for all $t$ $E[|\phi(X_t)|] < \infty$. Then $\phi(M_t)$ is a $\mathcal{F}_t$-submartingale.

b) Let $T_n$ be a point process with corresponding counting process $N_t$. Show that if $N_t$ is integrable, that is, $E[N_t] < \infty$ for all $t \geq 0$ then the point process is nonexplosive.

c) Let $N_t$ be a point process with $\mathcal{F}_t$-intensity $\lambda_t$. Prove that the following two conditions are equivalent.

- $N_\infty = \infty$ a.s.
- $\int_0^\infty \lambda_s \, ds = \infty$, a.s.