Problem 1:
Consider the stochastic control problem

\[ V(s, x) = \max_u E[(X_T)^r], \quad \text{where } 0 < r < 1 \]

subject to \( dX_t = a u dt + u dB_t, \quad X_s = x > 0, \)

where \( B_t \in \mathbb{R} \) is a Brownian motion, \( a \in \mathbb{R} \) is a constant, and

\[ T = \inf\{ t > s : X_t = 0 \} \land t_1, \]

where \( t_1 > s \) is a constant future time.

Solve this control problem and show that the optimal control is

\[ u^*(t, x) = \frac{a x}{1 - r}, \]

with corresponding optimal performance

\[ V(s, x) = x^r \exp\left( \frac{a^2(T - s)r^2}{2(1 - r)} \right). \]

Problem 2:
Consider the stochastic control problem

\[ V(s, x) = \sup_u E \left[ \int_s^\infty \exp(-\rho t) f(X_t) \, dt \right], \]

subject to \( dX_t = r u_t X_t \, dt + \alpha u_t X_t \, dB_t, \)

where \( r, \alpha, \) and \( \rho > 0 \) are constants, \( f(\cdot) \) is a bounded continuous function.

Assume that \( V \in C^2 \) and the the optimal Markov control \( u^* \) exists.

a) Write down the Bellman equation and show that

\[ \frac{\partial^2 V(t, x)}{\partial x^2} \leq 0. \]
b) Assume that $\frac{\partial^2}{\partial x^2} V(t, x) < 0$. Prove that

$$u^*(t, x) = -\frac{r \frac{\partial V(t, x)}{\partial x}}{\alpha^2 x \frac{\partial^2 V(t, x)}{\partial x^2}}$$

and that

$$2\alpha^2 \left( \exp(-\rho t) f(x) + \frac{\partial V(t, x)}{\partial t} \right) \frac{\partial^2 V(t, x)}{\partial x^2} - r^2 \left( \frac{\partial V(t, x)}{\partial x} \right)^2 = 0.$$ 

c) Assume that $\frac{\partial^2 V(t, x)}{\partial x^2} = 0$. Prove that $\frac{\partial V(t, x)}{\partial x} = 0$ and

$$\exp(-\rho t) f(x) + \frac{\partial V(t, x)}{\partial t} = 0.$$ 

d) Assume that $u^*_t = u^*(X_t)$ and that b) holds. Prove that $V(t, x) = \exp(-\rho t) g(x)$ where $g(x)$ satisfies

$$2\alpha^2 (f(x) - \rho g(x)) g''(x) - r^2 (g'(x))^2 = 0.$$ 

**Problem 3:**

Let $X_t$ denote your wealth at time $t$. Suppose that at any time $t$ you have a choice between two investments:

1) A risky investment where the unit price $P_1$ satisfies the equation

$$dP_1(t) = a_1 P_1(t) dt + \sigma_1 P_1(t) dB_t.$$ 

2) A safer (less risky) investment where the unit price $P_2$ satisfies

$$dP_2(t) = a_2 P_2(t) dt + \sigma_2 P_2(t) d\tilde{B}_t.$$ 

The parameters $a_i$ and $\sigma_i$ are constants such that

$$a_1 > a_2 \quad \text{and} \quad \sigma_1 > \sigma_2$$

and $B_t$ and $\tilde{B}_t$ are independent one-dimensional Brownian motions.

a) Let $u_t$ denote the fraction of the fortune $X_t$ which is placed in the risky investment at time $t$. Show that

$$dX_t = X_t (a_1 u_t + a_2 (1 - u_t)) dt + X_t (\sigma_1 u_t dB_t + \sigma_2 (1 - u_t) d\tilde{B}_t).$$

b) Assuming that $u$ is a Markov control, i.e., $u_t = u(t, X_t)$, find the generator of $(t, X_t)$. 

c) Write down the HJB equation for the stochastic control problem

\[ V(s, x) = \sup_u E \left[ \sqrt{T} \right], \]

where \( T = \min\{t > s : X_t = 0\} \wedge t_1. \)

d) Find the optimal control \( u^* \) for the problem in c).

**Problem 4:**
Consider the following control problem (*Bounded Velocity Follower*):

\[ V(x) = \min_\xi E \left[ \int_0^\infty \exp(-\alpha t) (x + B_t - \xi_t)^2 dt \right] \]

subject to \( -\infty < 0 < \xi_t \leq \theta_1 < \infty, \)

where \( B_t \) is a standard one-dimensional Brownian motion.

a) Write down the HJB equation and show that a bang-bang solution solves the problem.

b) Using the the smoothness fit assumption \( V(x) \in C^2 \) solve the control problem.

**Problem 5:**
Consider the following singular control problem.

\[ V(x) = \min_{\theta_t} \lim_{T \to \infty} \frac{1}{T} E \left[ \int_0^T \theta_t^2 dt + pU_T + rL_T \right] \]

subject to \( dX_t = -\theta_t dt + dB_t + dL_t - dU_t, \)

where the pair \( (U_t, L_t) \) is the two-sided regulator for the interval \([0, b]\). \( p \) and \( r \) are the cost of “hitting” the upper and lower boundaries \( x = b \) and \( x = 0 \), respectively.

Write down the HJB equation and find the optimal control policy \( \theta^* \).