Using Theory to Identify Transitory and Permanent Income Shocks: A Review of the Blundell-Preston Approach

October 12, 2004

1 Introduction

- Statements on changes in income inequality based on the cross-sectional distribution of income at two points in time can be very misleading: from the cross-section, one is unable to disentangle transitory from permanent components of the change in inequality.

- Two alternative approaches have been proposed in the literature to identify permanent and transitory income shocks:

  1. Agnostic approach— the shocks are separately identified using longitudinal income (i.e., wage or earnings) data for a panel of individuals/households. The autocovariance function of income contains information about the degree of persistence of shocks.

  2. Theory-based approach— according to the Permanent Income Hypothesis (hereafter, PIH), permanent shocks have a larger effect on consumption than transitory shocks, thus jointly analyzing data on changes in the income and the consumption distribution is informative for the nature of the shocks.
This is the approach pursued by Blundell and Preston (QJE, 1998). How far can one go with their methodology?

2 A Primer on the PIH

- Consider a household \((i)\) who lives for \(T\) periods, endowed with quadratic preferences

\[
U(c_{it}, c_{i,t+1}, . . . , c_{i,T}) = E_t \sum_{j=0}^{T-t} \beta^j \left[ ac_{i,t+j} - \left( \frac{b}{2} \right) c_{i,t+j}^2 \right],
\]

where \(E_t\) denotes the expectation on future income shocks (whose precise structure we will specify later) conditional on all the information available at time \(t\).

- The household can trade a riskless bond, but there is no insurance market for labor income risk. The period \(t\) budget constraint for this household is

\[
A_{i,t+1} = (1 + r_t) A_{it} + y_{it} - c_{it},
\]

where \(A_{it}\) denotes current wealth and \(y_{it}\) current income. There is a No-Ponzi-scheme terminal condition on wealth, but no borrowing constraint is imposed during the life of the agent.

- From the consumption Euler equation:

\[
a - bc_{it} = \beta E_t \left[ (1 + r_{t+1}) (a - bc_{i,t+1}) \right]
\]

- Assuming that \(\beta (1 + r) = 1\), we recover the random walk hypothesis, i.e.

\[
E_t c_{i,t+1} = c_{it}.
\]
• Iterating forward on the budget constraint (2), we obtain

\[
\frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^{j} E_t c_{i,t+j} = A_{it} + \frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^{j} E_t y_{i,t+j}
\]

\[
\frac{1}{1 + r} \left[ \frac{1 - \left( \frac{1}{1 + r} \right)^{T-t+1}}{1 - \frac{1}{1 + r}} \right] c_{it} = A_{it} + \frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^{j} E_t y_{i,t+j}
\]

\[
\frac{1 - (1 + r)^{-(T-t+1)}}{r} c_{it} = A_{it} + \frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^{j} E_t y_{i,t+j}
\]

where the LHS of the second row uses the random walk property, i.e.

\[ E_t c_{i,t+j} = c_{it}, \text{ for any } j \geq 0. \]

• Now define

\[
\Pi_{it} \equiv (A_{it} + H_{it}) \equiv A_{it} + \frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^{j} E_t y_{i,t+j}, \quad (3)
\]

as the value of permanent income, i.e. the sum of financial wealth \( A_{it} \) and human wealth \( H_{it} \), and to simplify the notation, define also the annuitization factor

\[
\rho_t = 1 - (1 + r)^{-(T-t+1)}. \quad (4)
\]

Note that as \( T \to \infty \), \( \rho_t = 1 \).

• Then, we can write

\[
\rho_t c_{it} = r \Pi_{it}, \quad (5)
\]

i.e., consumption equals the “annuity value” of permanent income.

• Now, use (3) to define the innovation (unexpected change) in permanent income at time \( t \) as

\[
\Pi_{i,t+1} - E_t \Pi_{i,t+1} = A_{i,t+1} - E_t (A_{i,t+1}) + \frac{1}{1 + r} \sum_{j=0}^{T-(t+1)} \left( \frac{1}{1 + r} \right)^{j} (E_{t+1} - E_t) y_{i,t+1+j},
\]

\[
= \frac{1}{1 + r} \sum_{j=0}^{T-(t+1)} \left( \frac{1}{1 + r} \right)^{j} (E_{t+1} - E_t) y_{i,t+1+j}, \quad (6)
\]

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where we have used the law of iterated expectations $E_{t} (E_{t+1}X_{t+1+j}) = E_{t}X_{t+1+j}$, and the fact that $A_{i,t+1} = E_{t} (A_{i,t+1})$, since there is no uncertainty at time $t$ about the evolution of wealth next period.

- Now, from (5) consider the innovation to consumption at time $t + 1$, or

\[
\rho_{t+1} [c_{i,t+1} - E_{t}c_{i,t+1}] = r [\Pi_{i,t+1} - E_{t}\Pi_{i,t+1}]
\]

\[
\rho_{t+1} \Delta c_{i,t+1} = \frac{r}{1 + r} \sum_{j=0}^{T-(t+1)} \left( \frac{1}{1 + r} \right)^{j} (E_{t+1} - E_{t}) y_{i,t+1+j}
\]

(7) where we have used the random walk property and equation (6).

- Moving (7) backward one period to time $t$, we arrive at

\[
\rho_{t} \Delta c_{it} = \frac{r}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^{j} (E_{t} - E_{t-1}) y_{i,t+j} \equiv \eta_{it},
\]

(8) which is the starting point of the Blundell-Preston’s article, i.e. their equation (8), explained in footnote 7 and footnote 8. This equation says that consumption growth between time $t - 1$ and $t$ is proportional to the revision in expected earnings due to the new information accruing in that same time interval.

3 Using the PIH to Identify the Nature of Income Shocks

- At this point, we need to make some assumptions on the labor income process (in levels). Blundell-Preston assume that labor income is the sum of two orthogonal components, a permanent component $y_{it}^{p}$ which follows a martingale, and a transitory component $u_{it}$ that is iid over time:

\[
y_{it} = y_{it}^{p} + u_{it},
\]

\[
y_{it}^{p} = y_{it}^{p,t-1} + v_{it}.
\]
Note that $v_{it}$ is the innovation to the permanent component. Both components are assumed to be $iid$ in the cross-section of households. Using $y_{it}^p = y_{it} - u_{it}$ from the first equation into the second equation, we obtain

$$y_{it} = y_{i,t-1} + v_{it} + u_{it} - u_{i,t-1}$$  \hspace{1cm} (9)

- With this structure of shocks in hand, we can express the RHS of (8) only as a function of the permanent innovation $v_{it}$ and the transitory innovation $u_{it}$. From (8):

$$\sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^j (E_t - E_{t-1}) y_{i,t+j} = (E_t - E_{t-1}) \left[ y_{it} + \frac{1}{1 + r} y_{i,t+1} + \left( \frac{1}{1 + r} \right)^2 y_{i,t+2} + \ldots \right]$$

(10)

- Using (9) into the first term of the RHS of (10):

$$E_t y_{it} - E_{t-1} y_{it} = y_{it} - (y_{i,t-1} - u_{i,t-1})$$

$$= (y_{i,t-1} + v_{it} + u_{it} - u_{i,t-1}) - (y_{i,t-1} - u_{i,t-1})$$

$$= v_{it} + u_{it}$$

In other words, the surprise in income $y_{it}$, compared to the one-step-ahead forecast $E_{t-1} y_{it}$, is the sum of permanent and transitory innovations at time $t$.

- Using (9) into the second term of the RHS of (10)

$$(E_t - E_{t-1}) y_{i,t+1} = (E_t - E_{t-1}) [y_{it} + v_{i,t+1} + u_{i,t+1} - u_{it}]$$

$$= (E_t - E_{t-1}) [(y_{i,t-1} + v_{it} + u_{it} - u_{i,t-1}) + (v_{i,t+1} + u_{i,t+1} - u_{it})]$$

$$= (E_t - E_{t-1}) [(y_{i,t-1} + v_{it} - u_{i,t-1}) + (v_{i,t+1} + u_{i,t+1})]$$

$$= v_{it},$$

since all the terms indexed by $t - 1$ drop out because $E_t (x_{i,t-1}) = E_{t-1} (x_{i,t-1}) = x_{i,t-1}$, and all the terms indexed by $t + 1$ drop out because $E_t (x_{i,t+1}) = E_{t-1} (x_{i,t+1}) = 0$. 

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• At this point, it is easy to see that
\[(E_t - E_{t-1}) y_{i,t+j} = v_{it} \quad \text{for any } j > 1\]

The forecast revision between \(t\) and \(t - 1\) about income beyond time \(t\) equals the permanent innovation at time \(t\).

• Going back to expression (8), the innovation to permanent income is
\[
\eta_{it} = r_1 + \frac{r}{1 + r} u_{it} + \frac{v_{it}}{1 + r} + \left( \frac{1}{1 + r} \right)^2 v_{it} + \ldots + \left( \frac{1}{1 + r} \right)^{T-t} v_{it}
\]
\[
= \frac{r}{1 + r} u_{it} + \left( \frac{r}{1 + r} \right) \left[ \frac{1 - \left( \frac{1}{1+r} \right)^{T-t+1}}{1 - \frac{1}{1+r}} \right] v_{it}
\]
\[
= \frac{r}{1 + r} u_{it} + \rho_t v_{it},
\]
where we have used the definition of the annuitization factor \(\rho_t\) given in (4).

• Now, let’s collect the two key equations we need for future reference
\[
c_{it} = c_{i,t-1} + v_{it} + \left( \frac{r}{1 + r} \right) \frac{1}{\rho_t} u_{it} \quad (11)
\]
\[
y_{it} = y_{i,t-1} + v_{it} + u_{it} - u_{i,t-1} \quad (12)
\]

• Compute now the growth in the cross-sectional variance of income for a cohort \(k\) between time \(t\) and \(t + 1\). From (12)
\[
\text{var}_{kt} (y) = \text{var}_{k,t-1} (y) + \text{var}_{kt} (v) + \text{var}_{kt} (u) + \text{var}_{k,t-1} (u) - 2 \text{cov}_{k,t-1} (y, u),
\]
\[
= \text{var}_{k,t-1} (y) + \text{var}_{kt} (v) + \text{var}_{kt} (u) + \text{var}_{k,t-1} (u) - 2 \text{var}_{k,t-1} (y, u),
\]
hence,
\[
\Delta \text{var}_{kt} (y) = \text{var}_{kt} (v) + \Delta \text{var}_{kt} (u),
\]
the growth in the variance of income is the sum of permanent inequality and the growth in uncertainty.
• Compute now the growth in the variance of consumption for a cohort \(k\) between time \(t\) and \(t+1\). From (11)

\[
\text{var}_{kt} (c) = \text{var}_{k,t-1} (c) + \frac{1}{\rho_t^2} \left( \frac{r}{1+r} \right)^2 \text{var}_{kt} (u),
\]

hence,

\[
\Delta \text{var}_{kt} (c) = \text{var}_{kt} (v) + \frac{1}{\rho_t^2} \left( \frac{r}{1+r} \right)^2 \text{var}_{kt} (u)
\]

which shows that the growth in the variance of consumption is dominated by the permanent component, as the term \([r/(1+r)]^2\) is second order. In particular, as \(r \to 0\), the RHS equals \(\text{var}_{kt} (v)\).

• Compute now the covariance between consumption and income for cohort \(k\) at time \(t\)

\[
E_k (c_{it} y_{it}) = E_k \left[ \left( c_{i,t-1} + \frac{1}{\rho_t} \eta_{it} \right) \left( y_{i,t-1} + v_{it} + u_{it} - u_{i,t-1} \right) \right]
\]

\[
E_k (c_{it} y_{it}) - E_k (c_{i,t-1} y_{i,t-1}) = E_k \left[ c_{i,t-1} (v_{it} + u_{it} - u_{i,t-1}) + \frac{1}{\rho_t} \eta_{it} (y_{i,t-1} + v_{it} + u_{it} - u_{i,t-1}) \right]
\]

\[
= E_k \left[ \frac{\eta_{i,t-1}}{\rho_{t-1}} (v_{it} + u_{it} - u_{i,t-1}) + \frac{\eta_{it}}{\rho_t} (y_{i,t-1} + v_{it} + u_{it} - u_{i,t-1}) \right]
\]

\[
= E_k \left[ \left( \frac{1}{\rho_{t-1}} \left( \frac{r}{1+r} \right) u_{i,t-1} + v_{i,t-1} \right) (v_{it} + u_{it} - u_{i,t-1}) \right.
\]

\[
+ \left( \frac{1}{\rho_t} \left( \frac{r}{1+r} \right) u_{it} + v_{it} \right) (y_{i,t-1} + v_{it} + u_{it} - u_{i,t-1}) \right]
\]

\[
= E_k \left[ - \frac{1}{\rho_{t-1}} \left( \frac{r}{1+r} \right) u_{i,t-1}^2 + \frac{1}{\rho_t} \left( \frac{r}{1+r} \right) u_{it}^2 + v_{it}^2 \right]
\]

\[
= \text{var}_{kt} (v) + \frac{r}{1+r} \Delta \left[ \frac{1}{\rho_t} \text{var}_{kt} (u) \right]
\]

• Now note that

\[
E_k (c_{it}) E_k (y_{it}) - E_k (c_{i,t-1}) E_k (y_{i,t-1}) = E_k \left( c_{i,t-1} + \frac{1}{\rho_t} \eta_{it} \right) E_k (y_{i,t-1} + v_{it} + u_{it} - u_{i,t-1})
\]

\[
- E_k (c_{i,t-1}) E_k (y_{i,t-1})
\]

\[
= 0
\]
Therefore, from the definition of the covariance, $\text{cov}_{kt}(c, y) = E_k(c_{it}y_{it}) - E_k(c_{it})E_k(y_{it})$, we obtain

$$\Delta \text{cov}_{kt}(c, y) = \text{var}_{kt}(v) + \frac{r}{\rho_t} \Delta \left[ \frac{1}{\rho_t} \text{var}_{kt}(u) \right],$$

which says that the growth in the covariance is also dominated by the permanent shock, since the latter is the common component between income and consumption.

Now, we are ready for the main result. Consider the following pair of difference in difference estimators

$$\Delta \text{var}_{kt}(y) - \Delta \text{var}_{kt}(c) = \left[ 1 - \frac{1}{\rho_t} \left( \frac{r}{1 + r} \right)^2 \right] \text{var}_{kt}(u) - \text{var}_{k,t-1}(u)$$

$$\Delta \text{cov}_{kt}(c, y) - \Delta \text{var}_{kt}(c) = \frac{r}{\rho_t (1 + r)} \left[ 1 - \frac{r}{\rho_t (1 + r)} \right] \text{var}_{kt}(u) - \frac{r}{\rho_{t-1} (1 + r)} \text{var}_{k,t-1}(u)$$

Note that, even if one takes a stand on $T$ and $r$, one cannot identify the variance of the transitory shocks year by year, unless an assumption is made on the initial condition.

However, if one is willing to assume that $T$ is large and $r$ is small, then

$$\Delta \text{var}_{kt}(y) - \Delta \text{var}_{kt}(c) \approx \Delta \text{var}_{kt}(u)$$

$$\Delta \text{cov}_{kt}(c, y) - \Delta \text{var}_{kt}(c) \approx 0$$

and the first difference allows us to identify the growth in the transitory component, while the second is a testable restriction provided by the theory.

See Figure V extracted from Blundell-Preston’s article: the growth in U.K. inequality was mostly transitory since the mid 1980s.

**Conclusions:** Neat contribution… The major drawbacks of the methodology are:

1. quadratic utility, which rules out precautionary behavior
2. complete absence of markets for risk-sharing (common assumption in this class of economies...)

3. $1 + r = 1/\beta$, which is not an equilibrium outcome in an economy with uninsurable risk and intertemporal trading through a riskless bond, at least if the borrowing constraint binds in some state. However, it looks like the analysis goes through even with $r$ fixed at a different level. However, if the variance of the shocks change, why should $r$ be invariant?

3.1 Individual-Specific Trend in Income

- Suppose that the income process is (as in Fatih’s paper)

\[
y_{it} = y^p_{it} + u_{it},
\]
\[
y^p_{it} = \alpha_i + y^p_{i,t-1} + v_{it},
\]

where $\alpha_i$ is an individual-specific expected growth in income. Then, it follows immediately that equation (11) is unchanged

\[
c_{i,t} = c_{i,t-1} + v_{it} + \left(\frac{r}{1+r}\right) \frac{1}{\rho_t} u_{it},
\]

and in particular $\alpha_i$ does not affect consumption growth because it is perfectly foreseen by households, but equation (12) becomes

\[
y_{it} = y_{i,t-1} + \alpha_i + v_{it} + u_{it} - u_{i,t-1}.
\]

- For $T$ large and $r$ small, it is easy to see that

\[
\Delta \text{var}_{kt}(c) \simeq \text{var}_{kt}(v)
\]
\[
\Delta \text{var}_{kt}(y) = \text{var}_{kt}(v) + \Delta \text{var}_{kt}(u) + \text{var}_{kt}(\alpha) + 2\text{cov}_{k,t-1}(\alpha, y)
\]
\[
\Delta \text{cov}_{kt}(c, y) \simeq \text{var}_{kt}(v) + 2\text{cov}_{k,t-1}(c, \alpha)
\]
where those two covariance components are clearly non-zero because $\alpha_i$ affects the level of both income and consumption.

- Consider now the moments we used earlier for identification:

$$\Delta \text{var}_{kt} (y) - \Delta \text{var}_{kt} (c) \simeq \Delta \text{var}_{kt} (u) + \var_{kt} (\alpha) + 2 \text{cov}_{k,t-1} (\alpha, y)$$

$$\Delta \text{cov}_{kt} (c, y) - \Delta \text{var}_{kt} (c) \simeq 2 \text{cov}_{k,t-1} (c, \alpha)$$

Hence, we lose both the identification of the shocks and the testable restriction of the theory.

- Conclusions: Fatih’s preferred model for income shocks is lethal to the Blundell and Preston identification strategy...

4 Identification with CARA Preferences

- Here, we relax the assumption on quadratic utility and study whether the identification results explained above hold in the case of CARA utility. We exploit Caballero’s (JME, 1990) closed-form solution of the dynamic consumption/saving problem of an agent with CARA preferences.

- Let’s modify the preferences in (1) to

$$U (c_{it}, c_{i,t+1}, ..., c_{i,T}) = E_t \sum_{j=0}^{T-t} \beta^j \left[ -\frac{1}{\theta} e^{-\theta c_{i,t+j}} \right] , \quad (13)$$

and postulate exactly the same income process for our agents.

- Assuming again $\beta (1 + r) = 1$, the Euler equation yields

$$e^{-\theta c_{it}} = E_t \left[ e^{-\theta c_{i,t+1}} \right] . \quad (14)$$
• Now, we guess that a solution to the above difference equation is a consumption process that is linear in levels (like in the quadratic case), i.e.

\[ c_{i,t+1} = \Gamma_{it} + \phi_{it} c_{it} + \xi_{i,t+1}, \tag{15} \]

where \( \Gamma_{it} \) is the expected slope of the consumption path and \( \xi_{i,t+1} \) is the innovation to consumption (or to permanent income), so by definition (from rational expectations...)

\[ E_t \xi_{i,t+1} = 0, \]

i.e. the surprise to consumption is uncorrelated over time. Note that \( \{ \Gamma_{it}, \phi_{it}, \xi_{i,t+1} \} \) are to be determined as part of the solution.

• Plugging (15) into the RHS of (14) gives

\[ e^{-\theta c_{it}} = E_t \left[ e^{-\theta (\Gamma_{it} + \phi_{it} c_{it} + \xi_{i,t+1})} \right] \Rightarrow e^{-\theta (1 - \phi_{it}) c_{it} + \theta \Gamma_{it}} = E_t \left[ e^{-\theta \xi_{i,t+1}} \right], \]

and note that if \( \phi_{it} \) were different from 1 then the Euler equation would pin down consumption regardless of the budget constraint, which is implausible. Thus, we must impose \( \phi_{it} = 1 \) and this allows us to obtain:

\[ \Gamma_{it} = \Gamma_t = \frac{1}{\theta} \ln E_t \left[ e^{-\theta \xi_{i,t+1}} \right] > \frac{1}{\theta} E_t \left[ -\theta \xi_{i,t+1} \right] = 0 \]

by Jensen’s inequality, which means that the slope of the consumption path is common across all individuals, and positive.

• Recall the intertemporal budget constraint (unchanged from the previous model):

\[ \frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^j E_t c_{i,t+j} = A_{it} + \frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^j E_t y_{i,t+j}, \]

and use (15) with \( \phi_{it} = 1 \) into this constraint to obtain:

\[
\begin{align*}
\left[ \frac{1 - (1 + r)^{-T-t+1}}{r} \right] c_{it} + \frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^j E_t \Gamma_{t+j} &= \Pi_{it}, \\
\rho_t c_{it} + \frac{r}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^j E_t \Gamma_{t+j} &= r \Pi_{it},
\end{align*}
\]

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where we have used the fact that $\xi_{it}$ is serially uncorrelated. Note that this last equation says that for a given amount of income uncertainty and for given permanent income, the optimal consumption is reduced, with respect to the quadratic utility case, due to precautionary savings (the second term on the LHS).

• Now, let’s follow exactly the same steps as for the earlier derivation with quadratic utility. Consider the innovation to consumption at time $t+1$, or

$$
\rho_{t+1} [c_{i,t+1} - E_t c_{i,t+1}] = r [\Pi_{i,t+1} - E_t \Pi_{i,t+1}] - \frac{r}{1+r} \sum_{j=0}^{T-(t+1)} \left( \frac{1}{1+r} \right)^j (E_{t+1} - E_t) \Gamma_{t+1+j}
$$

$$
\rho_{t+1} [c_{i,t+1} - \Gamma_t + c_{it}] = \frac{r}{1+r} \sum_{j=0}^{T-(t+1)} \left( \frac{1}{1+r} \right)^j (E_{t+1} - E_t) y_{i,t+1+j},
$$

where we have used equation (15) with $\phi_{it} = 1$ on the LHS, together with the fact that

$$
(E_{t+1} - E_t) \Gamma_{t+1+j} = \frac{1}{\theta} (E_{t+1} - E_t) \{\ln E_{t+1+j} \left( e^{-\theta \xi_{i,t+2+j}} \right) \} = 0
$$

for any $j > 0$, because $\xi_{it}$ is serially uncorrelated.

• Moving equation (16) backward one period to time $t$, we arrive at

$$
\rho_t \Delta c_{it} = \rho_t \Gamma_{t-1} + \frac{r}{1+r} \sum_{j=0}^{T-t} \left( \frac{1}{1+r} \right)^j (E_t - E_{t-1}) y_{i,t+j} = \rho_t \Gamma_{t-1} + \eta_{it},
$$

$$
\Delta c_{it} = \Gamma_{t-1} + \frac{1}{\rho_t} \eta_{it}
$$

(17)

where, comparing (15) to (17) we see immediately that

$$
\xi_{it} = \frac{1}{\rho_t} \eta_{it}, \text{ and } \Gamma_{t-1} = \frac{1}{\theta} \ln E_{t-1} \left[ e^{-\theta \eta_{it}} \right]
$$

Equation (17) corresponds to equation (17) in Blundell and Preston’s article, explained in Footnote 9.

• The key result is that since the slope of optimal consumption $\Gamma_t$ is common across all individual within a cohort, taking cross-sectional variances as we did in the previous section does not affect any of the conclusions about identification.
• **Conclusions:** The identification strategy remains valid with CARA utility.

5 Identification with CRRA Preferences or... when things turn ugly

• Here, we relax the assumption on quadratic utility and study whether the identification results above hold in the case of CRRA utility.

• There is no known tractable closed-form solution to the income-fluctuation problem in this case, hence one has to proceed by further *approximations* (and, as you will see, you need plenty...)

• Let’s modify the preferences in (1) to

\[
U (c_{it}, c_{i,t+1}, ..., c_{i,T}) = E_t \sum_{j=0}^{T-t} \beta^j \left[ \frac{c_{i,t+j}^{1-\gamma} - 1}{1 - \gamma} \right],
\]

(18)

where \(\gamma\) is the coefficient of relative risk aversion.

• From the Euler equation, imposing as usual \(\beta (1 + r) = 1\), we obtain

\[
c_{i,t-1}^{\gamma} = E_{t-1} [c_{it}^{-\gamma}]
\]

(19)

• Now, we guess that the optimal consumption rule specifies that the level of consumption is proportional to permanent income, i.e.

\[
c_{it} = \phi_{it} \Pi_{it},
\]

(20)

where \(\phi_{it}\) is allowed to depend on current and future moments of the income process, but it is assumed to be independent of past moments. Note that the guess is trivially true for \(t = T\), since optimality requires \(c_{iT} = \Pi_{iT}\). Also recall that the guess is true in the quadratic utility case, as in that case \(c_{it} = (r/\rho_t) \Pi_{it}\), for every \(t\).
• Define, as usual, the innovation to permanent income as

$$\xi_{it} \equiv \Pi_{it} - E_{t-1} \Pi_{it}. $$

Then, from the Euler equation (19) and the guess (20) of optimal consumption

$$c_{i,t-1}^{-\gamma} = E_{t-1} (\phi_{it} \Pi_{it})^{-\gamma} = E_{t-1} \{ \phi_{it} [E_{t-1} \Pi_{it} + \xi_{it}] \}^{-\gamma},$$

• Now, assume that $(\phi_{it}, \xi_{it})$ are independent, and take a second-order Taylor expansion of the RHS of the Euler equation around:

$$\bar{\xi}_{it} = 0 \Rightarrow \Pi_{it} = E_{t-1} \Pi_{it} \equiv \bar{\Pi}_{it}$$

$$\bar{\phi}_{it} = E_{t-1} \phi_{it}$$

• Define the function to be approximated,

$$f (\phi, \xi) = \{ \phi_{it} [\bar{\Pi}_{it} + \xi_{it}] \}^{-\gamma},$$

and recall that a second order approximation around $(\bar{\phi}, \bar{\xi})$ is:

$$f (\phi, \xi) \simeq f (\bar{\phi}, \bar{\xi}) + \bar{f}_1 (\phi - \bar{\phi}) + \bar{f}_2 (\xi - \bar{\xi}) + \frac{1}{2} \bar{f}_{11} (\phi - \bar{\phi})^2 + \frac{1}{2} \bar{f}_{22} (\xi - \bar{\xi})^2 + \frac{1}{2} \bar{f}_{12} (\xi - \bar{\xi}) (\phi - \bar{\phi}),$$

where we are neglecting terms higher or equal than third-order.

• Now, let’s compute the terms of the Taylor approximation, one at the time:

$$E_{t-1} \left[ f (\bar{\phi}, \bar{\xi}) \right] = (\bar{\phi}_{it} \bar{\Pi}_{it})^{-\gamma}$$

$$E_{t-1} \left[ \bar{f}_1 (\phi - \bar{\phi}) \right] = -\gamma \{ \bar{\phi}_{it} \bar{\Pi}_{it} \}^{-\gamma-1} \bar{\Pi}_{it} E_{t-1} (\phi_{it} - \bar{\phi}_{it}) = 0$$

$$E_{t-1} \left[ \bar{f}_2 (\xi - \bar{\xi}) \right] = -\gamma \{ \bar{\phi}_{it} \bar{\Pi}_{it} \}^{-\gamma-1} \bar{\phi}_{it} E_{t-1} (\xi_{it}) = 0$$

$$E_{t-1} \left[ \bar{f}_{11} (\phi - \bar{\phi})^2 \right] = \frac{\gamma (\gamma + 1)}{2} (\bar{\phi}_{it} \bar{\Pi}_{it})^{-\gamma} \frac{E_{t-1} (\phi_{it} - \bar{\phi}_{it})^2}{\bar{\phi}_{it}^2}$$

$$E_{t-1} \left[ \frac{1}{2} \bar{f}_{22} (\xi - \bar{\xi})^2 \right] = \frac{\gamma (\gamma + 1)}{2} (\bar{\phi}_{it} \bar{\Pi}_{it})^{-\gamma} \frac{E_{t-1} (\xi_{it} - \bar{\xi}_{it})^2}{\bar{\Pi}_{it}^2}$$

$$E_{t-1} \left[ \frac{1}{2} \bar{f}_{12} (\xi - \bar{\xi}) (\phi - \bar{\phi}) \right] = 0.$$
where the last line uses the assumption that \((\xi_{it}, \phi_{it})\) are independent.

- Collecting terms, we obtain the expression for the approximated Euler equation in the Appendix of Blundell-Preston’s article:

\[
\begin{align*}
    c_{i,t-1}^{-\gamma} & = E_{t-1} \left\{ \phi_{it} \left[ \bar{\Pi}_{it} + \xi_{it} \right] \right\}^{-\gamma} \\
    & \approx \left( \bar{\phi}_{it} \bar{\Pi}_{it} \right)^{-\gamma} \left\{ 1 + \frac{\gamma (\gamma + 1)}{2} \left[ \frac{\text{var}_{t-1} (\phi_{it})}{\phi_{it}^2} + \frac{\text{var}_{t-1} (\xi_{it})}{\Pi_{it}^2} \right] \right\}, \\
    & = \left( \bar{\phi}_{it} \bar{\Pi}_{it} \right)^{-\gamma} \left\{ 1 + \frac{\gamma (\gamma + 1)}{2} \left[ \frac{\text{var} (\phi_{it})}{\phi_{it}^2} + \frac{\text{var} (\xi_{it})}{\Pi_{it}^2} \right] \right\},
\end{align*}
\]

where the last step comes from the assumption that \(\phi_{it}\) is independent of past moments of the income process and from the fact that \(\xi_{it}\) is a forecast error, thus serially independent.

- Now, define

\[
K_{it} \equiv \frac{\gamma (\gamma + 1)}{2} \left[ \frac{\text{var} (\phi_{it})}{\phi_{it}^2} + \frac{\text{var} (\xi_{it})}{\Pi_{it}^2} \right], \tag{21}
\]

we arrive at the simpler expression for the Euler equation

\[
\begin{align*}
    c_{i,t-1}^{-\gamma} & \approx \left( \bar{\phi}_{it} \bar{\Pi}_{it} \right)^{-\gamma} [1 + K_{it}] \\
    c_{i,t-1} & \approx \left( \bar{\phi}_{it} \bar{\Pi}_{it} \right) [1 + K_{it}]^{-1/\gamma} = \bar{\phi}_{it} [\Pi_{i,t-1} (1 + r) - c_{i,t-1}] [1 + K_{it}]^{-1/\gamma},
\end{align*}
\]

where the last equality used the fact that

\[
\bar{\Pi}_{it} \equiv E_{t-1} \Pi_{it} = \Pi_{i,t-1} (1 + r) - c_{i,t-1}
\]

since:

\[
\begin{align*}
    \Pi_{i,t-1} & = A_{i,t-1} + \frac{1}{1 + r} \sum_{j=0}^{T-t-1} \left( \frac{1}{1 + r} \right)^j E_{t-1} y_{i,t-1+j} = A_{i,t-1} + \frac{y_{i,t-1}}{1 + r} + \left( \frac{1}{1 + r} \right)^2 E_{t-1} y_{it} + \ldots \\
    E_{t-1} \Pi_{it} & = E_{t-1} A_{it} + \left( \frac{1}{1 + r} \right) \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^j E_{t-1} y_{i,t+j} = A_{it} + \left( \frac{1}{1 + r} \right) E_{t-1} y_{it} + \ldots \\
    (1 + r) \Pi_{i,t-1} - E_{t-1} \Pi_{it} & = (1 + r) A_{i,t-1} + y_{i,t-1} - A_{it} = c_{i,t-1},
\end{align*}
\]

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where the last equality follows from the budget constraint.

• Note: this derivation is slightly different from what Blundell and Preston seem to have derived in the Appendix, but the difference is immaterial for the results.

• Collecting terms in (22), we arrive at

\[ c_{i,t-1} \approx \phi_{it}^{-1} \left( 1 + r \right) \left[ 1 + K_{it} \right]^{-1/\gamma} \frac{\Pi_{i,t-1}}{1 + \phi_{it}^{-1} \left[ 1 + K_{it} \right]^{-1/\gamma}}, \]

which verifies the initial guess of the optimal consumption rule, and yields implicitly an expression for \( \phi_{i,t-1} \):

\[ \phi_{i,t-1} = \phi_{it}^{-1} \left( 1 + r \right) \left[ 1 + K_{it} \right]^{-1/\gamma} \left( 1 + \phi_{it} \left[ 1 + K_{it} \right]^{-1/\gamma} \right)^{-1} \]

• Furthermore, from (20) and (22), we can write optimal consumption growth as

\[ \frac{c_{it}}{c_{i,t-1}} \approx \phi_{it} \left( 1 + r \right) \left[ 1 + K_{it} \right]^{1/\gamma}, \]

\[ \Delta \ln c_{it} \approx \frac{1}{\gamma} K_{it} + \ln \left( \frac{\phi_{it}}{\phi_{it}} \right) + \ln \left( \frac{\Pi_{it}}{E_{i-1} \Pi_{it}} \right), \]

\[ \Delta \ln c_{it} \approx \frac{1}{\gamma} K_{it} + \ln \left( \frac{\phi_{it}}{\phi_{it}} \right) + \ln \xi_{it}, \]

\[ \Delta \ln c_{it} \approx \Gamma_{it} + \ln \xi_{it}, \]  

(23)

where

\[ \Gamma_{it} \equiv \frac{1}{\gamma} \left\{ \frac{\gamma}{2} \left( \frac{\text{var} \left( \phi_{it} \right)}{\phi_{it}^2} + \frac{\text{var} \left( \xi_{it} \right)}{\Pi_{it}^2} \right) \right\} + \ln \left( \frac{\phi_{it}}{\phi_{it}} \right) \]

• The line above makes clear that with CRRA preferences the slope of consumption growth, in general, is not constant across individuals within a cohort. This is a problem for the identification strategy.

• Under what circumstances \( \Gamma_{it} = \Gamma_i? \)
First, note that the term $\bar{\phi}_{it}^2$ does not vary across individuals, since $\bar{\phi}_{it} = E_{t-1}\phi_{it}$ and $\phi_{it}$ does not depend on past moments of the income process.

Second, $\phi_{it}$ varies across individuals only as long as $K_{it}$ does.

Consider the expression for $K_{it}$ in (21): the two terms $\text{var}(\phi_{it}), \text{var}(\xi_{it})$ do not vary across individuals; the term $\bar{\phi}_{it}^2$ also does not vary across individuals, as explained. However, $\bar{\Pi}_{it} = E_{t-1}\Pi_{it}$ depends on the past individual income realizations.

Assume that $\text{var}(\xi_{it}) \simeq 0$, i.e. the variance of the forecast error in permanent income is small. Then, $K_{it}$ does not depend on $\bar{\Pi}_{it}$, and $\Gamma_{it} = \Gamma_t$.

It’s not over yet...we still need to do some more work on the last term in (23):

$$\ln \xi_{it} = \ln \left( \frac{\Pi_{it}}{E_{t-1}\Pi_{it}} \right) = \ln \left( \frac{\Pi_{it} + E_{t-1}\Pi_{it} - E_{t-1}\Pi_{it}}{E_{t-1}\Pi_{it}} \right) = \ln \left( 1 + \frac{\Pi_{it} - E_{t-1}\Pi_{it}}{E_{t-1}\Pi_{it}} \right)$$

$$\simeq \frac{\Pi_{it} - E_{t-1}\Pi_{it}}{E_{t-1}\Pi_{it}} = \frac{1}{E_{t-1}\Pi_{it}} \left[ \frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^j (E_t - E_{t-1}) y_{i,t+j} \right]$$

(24)

Now, consider the approximation of $\ln y_{i,t-j}$ around $\bar{y} = E_{t-1}y_{i,t+j}$

$$\ln y_{i,t-j} = \ln \bar{y} + \frac{1}{\bar{y}} (y_{i,t-j} - \bar{y}) \Rightarrow y_{i,t-j} = \bar{y} \ln y_{i,t-j} - \bar{y} (\ln \bar{y} - 1) \simeq \bar{y} \ln y_{i,t-j},$$

where the last approximate equality comes from ... (don’t ask me).

Substituting $y_{i,t-j} \simeq (E_{t-1}y_{i,t+j}) \ln y_{i,t-j}$ into equation (24), one arrives at

$$\ln \xi_{it} \simeq \frac{1}{E_{t-1}\Pi_{it}} \left[ \frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^j E_{t-1}y_{i,t+j} (E_t - E_{t-1}) \ln y_{i,t+j} \right]$$

$$\simeq \frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^j (E_t - E_{t-1}) \ln y_{i,t+j},$$

(25)

where the last line holds if, according to Blundell and Preston (page 634),

$$\frac{1}{E_{t-1}\Pi_{it}} \left[ \frac{1}{1 + r} \sum_{j=0}^{T-t} \left( \frac{1}{1 + r} \right)^j E_{t-1}y_{i,t+j} \right] \simeq 1.$$
• Now we need to assume that the process for log income is the same as the one for income levels in (12),

\[ \ln y_{it} = \ln y_{i,t-1} + v_{it} + \Delta u_{it}, \]

and using this process into (25), we obtain the familiar expression for consumption growth

\[ \Delta \ln c_{it} \simeq \Gamma_t + v_{it} + \frac{1}{\rho_t} \left( \frac{r}{1+r} \right) u_{it}, \]

that together with the process for income yields the standard identification approach.

• Conclusions:

- with CRRA utility, the approximations to the true model needed to maintain validity of the original identification strategy are very courageous, at the very least...

- one could run a “Monte-Carlo experiment” on a buffer-stock OLG model where households have CRRA preferences. The \( var_{k,t}(u) \) are time-varying parameters of the model, so by simulation of histories of cohorts one can verify if the Blundell-Preston identification strategy works satisfactorily in practice

- I am convinced that there is scope for improvement...

6 An Alternative Theory-based Approach to Identification

• This is based on work I am doing with Jonathan Heathcote and Kjetil Storesletten

• We have developed a framework where we can separately identify two types of shocks to wages: “insurable” and “uninsurable”. Hence, we allow for the existence of markets for partial risk-sharing....
• The framework includes households with CRRA preferences, endogenous labor supply, heterogeneity in taste for leisure

• We can derive exact closed-form expressions for all cross-sectional variances and covariances, only as a function of the variances of the shocks and the deep preference parameters

• The approach also allows us to identify the preference parameters

• What is the trick? Actually, two tricks are needed...

  – Log Normality of innovations to insurable and uninsurable shocks

  – We focus on a sort of “Constantinides-Duffie” equilibrium where the interest rate takes the value where the intertemporal dissaving motive is exactly offset by the precautionary motive, hence households do not carry around any wealth across periods...
transitory components by comparison with older cohorts when they were at a similar age but that this is also true for permanent income inequality. As the variance of permanent shocks does not appear to have risen for any cohort, the implication is that this difference reflects an increase in initial permanent income inequality for younger cohorts. The explanation for a growth in overall consumption inequality can therefore be attributed in part to an aging population and in part to new cohorts facing higher levels of initial income inequality.