Part I. Econometric Theory

1. We have examined in great detail the least squares estimator of the parameter vector $\beta$ in the linear regression model, $E[y|x] = x'\beta$. We have also encountered the least absolute deviations estimator, which is an estimator of the parameter vector $\beta$ in the model $\text{Med}(y|x) = x'\beta$. Rather than use least squares, we use linear programming methods to compute the estimator of $\beta$. Suppose the model is changed to a “quantile regression” which states $Q(y|x, \alpha) = x'\beta(\alpha)$, where $\alpha$ is a specific quantile. For example, the case of $\alpha = .5$ is the median regression just noted. The quantile regression estimator can be computed for any $\alpha$; it is done by linear programming methods. Discuss how the researcher will obtain an estimator of the asymptotic covariance matrix for this estimator.

2. Suppose $b$ is an estimator of a regression coefficient vector. The asymptotic covariance matrix for $b$ is $V$. Now, we are going to compute a function of $b$, $g(b)$. The delta method and Krinsky and Robb are two methods of estimating the asymptotic variance of $g(b)$. Describe these two methods in detail, and distinguish between them.

3. Prove that the least squares estimator of $\beta$ in the linear regression model is consistent. What assumptions are needed to complete your proof?
Part II. Applications

The following exercises use the gasoline data which we have used at various points before. The data are posted on the course web page in the file gasoline.csv.

All variables in all regressions discussed below are assumed to be logarithms. As such, in the results to follow, all estimated coefficients are estimates of elasticities. In the following,

\[ lg = \log(g), \ lpg=\log(pg), \ ly=\log(pcincome), \ lpnc=\log(pnc) \text{ etc.} \]

1. Consider a nonlinear regression model of the form

\[ lg = \beta_1 + \beta_2 \ lpg + \beta_3 \ ly + \beta_4 \ ly^2 + \varepsilon. \]

Use linear least squares to estimate the coefficients of the model. The marginal effect of \( ly \) on \( \text{E}[lg|lp,ly] \) is

\[ \frac{\partial \text{E}[lg|lp,ly]}{\partial y} = \beta_3 + 2\beta_4 ly, \]

which depends on \( ly \). Form a confidence interval for this marginal effect at income \( = 15,912.7 \), so \( ly = 9.67487 \) (the mean) [Note, the estimate is of the form \( w_1b_3 + w_2b_4 \) where \( w_1=1 \) and \( w_2= 9.67487 \).] You will need to estimate the variance of this statistic using your regression results. (If you are using LIMDEP, include :Covariance in your regression command to display the matrix you need. You can access this matrix from the project window in the Matrices list – it is VARB. If you are using some other program, use whatever command you need to display the covariance matrix for your regression coefficient estimates.)

2. Standard error for a nonlinear function of a parameter: In some contexts (such as in binary choice modeling), the function \( c(\beta) = \exp(\beta) - 1 \) is an interesting quantity. (It is the 'odds ratio' in a logit model.) We will consider this function for the price elasticity in our gasoline. In the gasoline consumption model, we consider \( c(\beta_2) = \exp(\beta_2)-1. \)

a. Derive the asymptotic variance of \( c(\beta_2) \) using the delta method.

b. Using your regression results from part 1, compute the estimate, \( c(\hat{b}_2) - 1 \) and the asymptotic variance using your result in part a. (You can do this with a hand calculator if need be.)

c. We will now use Krinsky and Robb to estimate the standard error of \( c(\hat{b}_2) \). Your estimate of the parameter \( b_2 \) is given with the regression results. (Hint, it is -0.14609.) You also have an estimate of the asymptotic standard error in the regression results. (Hint, it is 0.02268.) To apply the K&R method for this setting, you need only do the following steps:

   (1) draw a random sample of \( N \) observations from the normal population with mean -0.14609 and standard deviation 0.02268. (Use \( N=1000. \) If you are using nlogit, just SAMPLE;1-1000$ then CREATE;b2i=Rnn(-.14609,.02268)$ If you are using a different package that allows you to draw standard normal values, \( w(i) \), just compute -.14609+.02268*w(i).

   (2) for each value of \( b2i \), compute \( ci = \exp(b2i)-1 \). Now, just use the descriptive statistics feature of your program to get the standard deviation of your set of draws. How does the answer you obtain compare to the results in part b?

3. Standard error for nonlinear function of 2 parameters. The partial effect with respect to \( ly \) is a function of \( ly \). Indeed, with \( b_3 = 3.87531 \) and \( b_4 = -.18677 \), we can see that holding \( lpg \) fixed at anything (such as a middling value of 1.5), the function is a quadratic in \( ly \). With the first term positive and the quadratic term negative, it is a parabola, as you can see in the figure below.
Notice that the function reaches a maximum at $\mathsf{ly} = \text{about 10.3}$. We can find this more precisely by noting the top of the hill occurs where the partial effect with respect to $\mathsf{ly}$ equals zero. That is,

$$
\frac{\partial \mathcal{E}[\mathsf{lg} \mid \mathsf{lp}, \mathsf{ly}]}{\partial \mathsf{ly}} = \beta_3 + 2\beta_4 \mathsf{ly} = 0, \text{ or } \mathsf{ly}^\ast = \frac{\beta_3}{(-2\beta_4)}
$$

We will use our estimation results to estimate $\mathsf{ly}^\ast$ and estimate the asymptotic variance. Suppose that the estimators are $b_3$ and $b_4$ and the elements of the asymptotic covariance matrix are $v_{33}$, $v_{34}$ and $v_{44}$. Using the delta method, derive the asymptotic variance for the estimator of $\mathsf{ly}^\ast$.

4. The following exercise is based on the Christensen and Greene electricity data. We will use a simple unnormalized cost function

(a) \[ \log c = \beta_1 + \beta_2 \log q + \beta_3 (1/2)\log^2 q + \beta_4 \log pc + \beta_5 \log pl + \beta_6 \log pf + \varepsilon \]

The model implies a “U” shaped cost curve. Average cost is $c/q$

(b) \[ \frac{c}{q} = \exp(\log(c/q)) = \exp[\beta_1 + (\beta_2 - 1)\log q + \beta_3 (1/2)\log^2 q + \beta_4 \log pc + \beta_5 \log pl + \beta_6 \log pf] \]

At least for certain values of $\beta_2$ and $\beta_3$ this will produce a U shaped average cost curve. As a preliminary for this exercise, I am going to fit the regression in (a) by least squares. Then, when I have the coefficients in hand, I am going to compute the value of $c/q$ in (b) for a range of values of $\log q$, while holding the price variables at the logs of the mean values. The script appears at the end of this problem set, with the others. The result of the simulation based on the least squares results is
It looks like the theory works. The value of logq at which the cost curve reaches its minimum appears to be about 9.3. That means the output at which the average cost curve bottoms out (this is called the “efficient scale”) is \(\exp(9.3)\) or about 10,938. The range of outputs in the sample is 4 to 72,000, however 11,000 is about the median.

The actual output at which the average cost curve reaches its minimum is that at which the “scale elasticity,” \(\frac{\partial \log c}{\partial \log q}\) equals 1. Since

\[
\frac{\partial \log c}{\partial \log q} = \beta_2 + \beta_3 \log q = 1,
\]

the actual value we seek is

\[
q^* = \exp\left(\frac{1-\beta_2}{\beta_3}\right)
\]

Your assignment for this exercise is

(1) using your least squares results, obtain the estimate of \(q^*\). (Hint, the value you obtain will differ noticeably from the value shown in the picture. The reason is that the picture is computed just by simulating the cost curve at the average prices, not by manipulating the regression coefficients.

(2) Obtain an appropriate asymptotic standard error for your estimate of \(q^*\) by using the delta method.

(3) Compute a confidence interval for your estimate based on your results of (1) and (2).

5. (This exercise requires a fairly complicated use of your software – it may not be available. If you do not have this capability, just describe how you would proceed if you could.) The asymptotic standard errors for the least squares estimates obtained in part 1 are the square roots of the diagonal elements of \(s^2(X'X)^{-1}\). Bootstrapping is an alternative way to obtain asymptotic standard errors. Compute a bootstrapped covariance matrix for the OLS estimator in part 1 and report your results. Compare them to the conventional estimator. (nlogit code for this computation appears at the end of the next page.)
import; file=h:\gasoline.csv
create ; lg=log(g); lpg=log(pg); ly=log(pcincome); lpc=log(pnc); lppt=log(ppt)
reg; lhs=lg; rhs=one, lpg, ly, ly*ly; covariance
calc ; v33=varb(3,3); v44=varb(4,4); v34=varb(3,4)
calc ; lybar=xbr(ly)
calc ; list ; sdv = sqr(v33 + 2*(2*lybar)*v34 + (2*lybar)^2 * v44)
calc ; list ; pe = b(3) + 2*lybar*b(4)
calc ; list ; lowerci = pe - 1.96*sdv; upperci = pe + 1.96*sdv
partials; effects: ly ; means
? given b2, c(b2)=exp(b2)-1. The derivative is exp(b2), so the variance
? is [exp(b2)]^2 times the variance.
calc ; list ; b2 = b(2); c = exp(b2)-1
(calc ; list ; var = (exp(b2))^2 * varb(2,2)
calc ; list ; sd = sqr(var)
wald ; fn1 = exp(b_lpg)-1
? Now, Krinsky and Robb
Sample ; 1 - 1000
Calc ; sd2 = sqr(varb(2,2))
create ; b2l = Rnn(b2, sd2)
create ; ci = exp(b2l)-1
wald ; fn1 = exp(b_lpg)-1
? Part 3
sample; 1-52
reg; lhs=lg; rhs=one, lpg, ly, ly*ly
sample; 1
fplot; fcn=-18.6042-.14609*1.5+3.87531*lyv-.18677*lyv*lyv
; vary(lyv) ; pts=200 ; limits=9,11 ; labels=lyv; start=9.5
sample; 1-52
partials; effects: ly & ly=9(.1)11
calc ; b3=b(3); b4 = b(4); v33=varb(3,3); v44=varb(4,4); v34=varb(3,4)
calc ; lystar = b3/(-2*b4)
calc ; vlystar = g3^2*v33 + g4^2*v44 + 2*g3*g4*v34
calc ; list ; lystar ; sd = sqr(vlystar)
wald ; labels = 4_bh ; start=b ; varb ; fn1 = exp(b_lpg)-1
Part 4.
import; file="H:\Courses-Stern\Econometrics\Problems\DataFiles\electricity.csv"
? Set up the data for the regression
create ; logq=log(output)
; logpc=log(cprice) ; logpf=log(fprice) ; logpl=log(lprice)
create ; logc = log(cost) ; logq2 = .5*logq*logq
? Least squares regression
regress; lhs=logc; rhs=one, logq, logq2, logpc, logpf, logpl
? Explicit computation of the cost function at data means, demonstration.
calc ; meanlpc = log(xbr(cprice))
; meanlpf = log(xbr(fprice))
; meanlpl = log(xbr(lprice))
calc ; b1=b(1); b2=b(2); b3=b(3); b4=b(4); b5=b(5); b6=b(6)
sample; 1
fplot ; fcn=exp(b1 + (b2-1)*lq + b3*lq*lq*.5 + b4*meanlpc + b5*meanlpf + b6*meanlpl)
; start = 5 ; limits = 2,15 ; pts=200 ; labels=1q
plot(lq)
? Compute efficient scale. Explicitly using theoretical results
calc; list; qstar=exp((1-b2)/b3)
calc ; g2 = qstar * (-1/b3)
calc ; g3 = qstar * (-log(qstar))/b3
(calc ; v22 = varb(2,2); v33 = varb(3,3); v23 = varb(2,3)
calc ; list ; sd = sqr(g2^2*v22 + g3^2*v33 + 2*g2*g3*v23)
calc ; list ; lower = qstar - 1.96*sd
; upper = qstar + 1.96*sd 
? The same computation using the built in program function
wald ; start=b
    ; var=varb
    ; labels=6_c
    ; fn1=exp((1-c2)/c3)

? Part 5.
sample;1-52$
procedure $
regress ; quietly ; lhs = lg ; rhs=one,lpg,ly,ly^2$
endproc $
exec ; n=50 ; bootstrap=b$
regress ; lhs = lg ; rhs=one,lpg,ly,ly^2