1. The distribution of the number of spots showing on the upturned face of a thrown die is discrete uniform, $X = 1, 2, 3, 4, 5, 6$ each with probability $\frac{1}{6}$. Find the mean and variance of $X$. Let $Y = X_1 + X_2$ be the sum of the values shown by two dice thrown independently. Find the probability distribution, mean and variance of the random variable $Y$.

The possible outcomes for $X_1 + X_2$ are $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$. The probabilities can be found by arranging the outcomes in a $6\times6$ grid and counting the number of ways each outcome occurs. The probabilities are:

<table>
<thead>
<tr>
<th>$Y$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
<th>$9$</th>
<th>$10$</th>
<th>$11$</th>
<th>$12$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(Y)$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\frac{4}{36}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{6}{36}$</td>
<td>$\frac{5}{36}$</td>
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<td>$\frac{2}{36}$</td>
<td>$\frac{1}{36}$</td>
</tr>
</tbody>
</table>

$$E[Y] = \sum y \cdot P(Y=y) = (2+6+12+20+\ldots)$$

It's easier to note that $E[Y] = E[X_1] + E[X_2] = 3.5 + 3.5 = 7$.
So just find the variance of $X_i = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6} - (\frac{21}{6})^2 = \frac{91}{6} - \frac{12.25}{6} = 15.167 - 12.25 = 2.917$. Adding the two gives $5.833$.

2. The random variable $X$ has continuous uniform distribution with $0 < X < 1$. What is the density function of $X$? What is the CDF? Derive the mean and variance of $X$.

$$f(x) = 1, \ 0 \le x \le 1. \ F(x) = \int_0^x 1 dt = x |_0^1 = x - 0 = x. \ E[x] = \int_0^1 x \times 1 dx = \frac{x^2}{2} |_0^1 = 1 - 0 = \frac{1}{2}$$

$$E[x^2] = \int_0^1 x^2 \times 1 dx = \frac{x^3}{3} |_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3} \ so \ Var[X] = (1/3) - (1/2)^2 = 1/12$$

3. Referring to the uniformly distributed random variable in 2, what is the conditional density for $X|X > .3$? What are the conditional mean and variance of this random variable?

The density of $X|X > .3$ is $f(x)/Prob(X>.3) = 1/7$.
This is a uniformly distributed variable over the range $[a,b]$ where $a = .3$ and $b = 1$.
The mean as always is the midpoint, which is .65.
$Var[X|X>.3] = E[X^2|X>.3] - \{E[X|X>.3]\}^2 = \frac{1}{12} - \frac{12}{12} = -\frac{11}{12}$
\[ \int_0^1 x^2 \frac{1}{7} \, dx = \frac{1}{7} \int_0^1 x^2 \, dx = \frac{1}{7} \left( \frac{1}{3} \cdot \frac{1}{0.27} \right) = \frac{0.973}{2.1} = .4633 \quad \text{So, the variance is } .4633 - .652 = .0408. \]

   a. \[ E[X] = \int_0^1 x \times 2x \, dx = 2 \int_0^1 x^2 \, dx = 2 \left( \frac{1}{3} \right) = \frac{2}{3} \]
   b. \[ Y = X^2 \] so \[ X = Y^{1/2} \] and \[ J = dX/dY = \frac{1}{2} Y^{-1/2}. \] \[ f(y) = 2 \, y^{1/2} \left( \frac{1}{2} \right) y^{-1/2} = 1, \quad 0 \leq y \leq 1. \] \[ E[y] = \frac{1}{2} \] from our earlier results.
   c. By theorem A in 4.1.1, we get \[ E[Y] \] by finding \[ E[X^2] \] directly.
   \[ E[X^2] = \int_0^1 x^2 \times 2x \, dx = 2 \int_0^1 x^3 \, dx = 2 \left( \frac{1}{4} \right) = \frac{1}{2}. \] \[ QED. \]
   d. By the definition,
   \[ \text{Var}[X] = \int_0^1 2x(x - 2 / 2)^2 \, dx = \int_0^1 2x(x^2 - (4/3)x + 4/9) \, dx = \int_0^1 (2x^3 - (8/3)x^2 + (8/9)x) \, dx \]
   \[ = \left[ \frac{2}{4} x^4 \right]_0^1 - \left[ \frac{8}{3} x^3 \right]_0^1 + \left[ \frac{8}{9} x^2 \right]_0^1 - \frac{8}{9} + \frac{8}{9} \cdot \frac{2}{36} \]
   \[ = \frac{1}{2} - \frac{1}{9} + \frac{1}{36} = \frac{18}{36} - \frac{16}{36} = \frac{2}{36} = \frac{1}{18}. \]
   By Theorem B, \[ \text{Var}[X] = E[X^2] - (E[X])^2 = \frac{1}{2} - \left( \frac{2}{3} \right)^2 = 18/36 - 16/36 = 2/36. \]
   \[ QED. \]

5. Suppose \( X \) is exponentially distributed, \( f(x) = \alpha \exp(-\alpha x) \) where \( \alpha = 0.1 \). How would you use a random number generator to generate a sample of observations from this population?

We did this exercise in class. See slide 79, “Generating Random Samples.”

6. The random variables \( Y \) and \( X \) have a bivariate normal distribution with means 1 and 2, variances 2 and 5 and covariance 3. Compute the slope and intercept in the conditional mean function \( E[y|x] \). Compute the squared correlation between \( y \) and \( x \). Compute the squared correlation between \( y \) and \( E[y|x] \).

The intercept in \( E[y|x] \) is \( E[Y] - \beta E[X] \) where the slope is \( \beta = \text{Cov}(x,y)/\text{Var}[x] = 3/5 \). The intercept is therefore \( 1 - (3/5)2 = 1 - 6/5 = -1/5 \). The correlation is \( \text{Cov}(x,y)/\text{Sqr}(\text{Var}[x]\text{Var}[y]) = 3/\text{Sqr}(2(5)) = 3/\text{Sqr}(10) \). The squared correlation is 9/10. The squared correlation between \( Y \) and \( X \) is the same as the squared correlation between \( Y \) and \( a + bx \).

7. Consider the joint distribution of two random variables, \( Y \), which is the number of failures of some component (disk drive) in a brand of computer per unit of time and \( X \), the average lifetime of some different but related component (a chip). Note that \( Y \) is a discrete random variable and \( X \) is a continuous random variable. Suppose that the conditional distribution of \( y|x \) is

\[ f(y|x) = e^{\beta x}(\beta x)^y / y!, \quad y = 0,1,\ldots, x \geq 0, \beta > 0, \]

while the marginal distribution of \( x \) is

\[ f(x) = \theta e^{\theta x}, \quad x \geq 0, \theta > 0. \]

Thus, conditioned on \( x, y \) has a Poisson distribution with parameter \( \beta x \), while \( x \), unconditionally, has an exponential distribution.

a. What is the joint distribution of these two random variables, \( f(y,x) \)?

\[ f(y,x) = f(y|x)f(x) = \left( e^{\beta x}(\beta x)^y / y! \right) \theta e^{\theta x} = \theta e^{(\beta + \theta)x} (\beta x)^y / y! \]
b. Show that the unconditional density of $y$ is $f(y) = \delta (1-\delta)^y$ where $\delta = \theta/(\beta + \theta)$.

To find the unconditional marginal density of $y$, we need to integrate $x$ out of the joint density.

$$f(y) = f(y) = \int_0^\infty \theta e^{-(\beta+\theta)x} (\beta x)^y / y! \, dx = \frac{\theta \beta^y}{y!} \int_0^\infty e^{-(\beta+\theta)x} x^y \, dx$$

This is a gamma integral, as we discussed in class. $\int_0^\infty e^{-(\beta+\theta)x} x^y \, dx = \frac{\Gamma(y+1)}{(\beta + \theta)^{y+1}}$.

Combining terms, the density is

$$\frac{\Gamma(y+1)}{(\beta + \theta)^{y+1}} \frac{\theta \beta^y}{y!} = \frac{\theta \beta^y}{(\beta + \theta)^{y+1}} \frac{\Gamma(y+1)}{(\beta + \theta)^{y+1}} = \delta (1-\delta)^y.$$  

Accordingly,

$$\frac{\Gamma(y+1)}{(\beta + \theta)^{y+1}} \theta \beta^y = \frac{\Gamma(y+1)}{(\beta + \theta)^{y+1}} \frac{\theta \beta^y}{(\beta + \theta)^{y+1}} = \delta (1-\delta)^y.$$  

This is the distribution of $y$. It follows then, that in our conditional distribution, $E[y|x] = Var[y|x] = \beta x$. You could obtain $E[y]$, $Var[y]$, and $Cov[y,x]$ from the marginal distribution $f(y)$ and the joint distribution $f(x,y)$ by summing and integrating using the definitions. But, there is a much easier way.

c. Show that $E[x] = 1/\theta$ and $Var[x] = 1/\theta^2$.

$$E[X] = \int_0^\infty x \theta e^{-\theta x} \, dx = \theta \frac{\Gamma(2)}{\theta^2} = \frac{1}{\theta},$$

$$E[X^2] = \int_0^\infty x^2 \theta e^{-\theta x} \, dx = \theta \frac{\Gamma(3)}{\theta^3} = \frac{2}{\theta^2}.$$  

$$Var[X] = \frac{2}{\theta^2} - \left( \frac{1}{\theta} \right)^2 = \frac{1}{\theta^2}.$$  

If $Z$ has a Poisson distribution with parameter $\alpha$, then, $f(z) = e^{\alpha z}/z!$, $E[z] = Var[z] = \alpha$. It follows then, that in our conditional distribution, $E[y|x] = Var[y|x] = \beta x$. You could obtain $E[y]$, $Var[y]$, and $Cov[y,x]$ from the marginal distribution $f(y)$ and the joint distribution $f(x,y)$ by summing and integrating using the definitions. But, there is a much easier way.

d. Using the fundamental results:

$$E[y] = E[E[y|x]],$$

$$Var[y] = E[Var[y|x]] + Var[E[y|x]],$$

$$Cov[x,y] = Cov[x, E[y|x]],$$

show that $E[y] = \beta/\theta = \gamma$, $Var[y] = \beta/\theta + (\beta/\theta)^2 = \gamma(1+\gamma)$, and $Cov[x,y] = \beta/\theta^2 = \gamma/\theta$.

The distribution of $y|x$ is Poisson with mean = variance = $\beta x$. Therefore,

$$E[y] = E_x E[y|x] = E_x E[x] = \beta \times 1/\theta.$$  

$$Var[y] = E[Var[y|x]] + Var[E[y|x]] = E[\beta x] + Var[\beta x] = \beta/\theta + \beta^2 (1/\theta^3)$$

$$Cov[x,y] = Cov[x, E[y|x]] = Cov[x, \beta x] = \beta Var[x] = \beta / (\theta^2).$$  

QED.