Excess Returns and Beta: Deriving the Security Market Line

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I. We showed that market forces combined with a search by investors for efficient portfolios would produce the following relationships for each security \((i, j = 1, \ldots, n)\) in a portfolio:

\[
\frac{R_i - R_f}{\partial \sigma_m / \partial X_i} = \frac{R_m - R_f}{\sigma_m}
\]

This expression can be rewritten as

\[
R_i = R_f + \left( \frac{R_m - R_f}{\sigma_m} \right) \partial \sigma_m / \partial X_i
\]

II. Expression (2) says that the equilibrium expected return \(R_i\) on security \(i\) should equal the risk-free rate \(R_f\) plus the market price of risk \((R_m - R_f) / \sigma_m\) times \(\partial \sigma_m / \partial X_i\).

What is \(\partial \sigma_m / \partial X_i\)? It is the increase in risk \(\sigma_m\) associated with a small increase in asset \(i\), in other words, it is the risk contribution of security \(i\) to portfolio risk, \(\sigma_m\). We can derive \(\partial \sigma_m / \partial X_i\) by taking the derivative of the expression for the variance of a portfolio with respect to \(X_i\). The result, as shown in Garbade (p. 175, fn 12) is:

\[
\frac{\partial \sigma_m}{\partial X_i} = \frac{1}{\sigma_m} \sum_{j=1}^{n} X_j \text{Cov}(R_i, R_j)
\]

This expression states that the risk contribution of a security to the portfolio depends on the covariance of asset \(i\) with each and every other asset. This makes considerable sense because we know that only systematic risk matters in a portfolio and systematic risk is measured by covariance.
III. The problem with expression (3) above is that it is not operational; there are too many covariances needed to measure the risk of security $i$. We can simplify the measurement problem by recalling the regression equation (security characteristic line) relating the return on an individual security, $R_i$, to the return on an index, $R_m$, consisting of all other securities in the market. In particular, we have:

$$R_i = \alpha_i + \beta_i R_m + e_i$$

Note that $R_m$ is defined as the weighted average return on all securities in the market:

$$R_m = \sum_{j=1}^{n} X_j R_j.$$

From statistics we know that the definition of the regression coefficient in expression (4) is given by:

$$\beta_i = \frac{\text{Cov}(R_i, R_m)}{\sigma_m^2}$$

This means that we can take the expression for $R_m$ in (5) and substitute it into expression (6) to get the following:

$$\beta_i = \frac{\text{Cov}(R_i, \sum_{j=1}^{n} X_j R_j)}{\sigma_m^2}$$

Expression (7) can be simplified as follows:

$$\beta_i = \frac{\sum_{j=1}^{n} X_j \text{Cov}(R_i, R_j)}{\sigma_m^2}$$

$$\beta_i = \frac{1}{\sigma_m} \left[ \frac{1}{\sigma_m} \cdot \sum_{j=1}^{n} X_j \text{Cov}(R_i, R_j) \right]$$
Notice that the expression inside the brackets in (9) is identical to the expression for $\partial \sigma_m / \partial X_i$ in (3). Thus, we can use expression (3) to rewrite (9), as follows:

$$\beta_i = \frac{1}{\sigma_m} \cdot \frac{\partial \sigma_m / \partial X_i}{\sigma_m}$$

IV. Finally, we can produce the result we want. We can use expression (10) to simplify expression (2), which is where we started, and make it operational. From (10) we have

$$\frac{\partial \sigma_m / \partial X_i}{\sigma_m} = \beta_i \sigma_m$$

We can then substitute (11) into (2), which produces

$$R_i = R_i + \left( \frac{R_m - R_f}{\sigma_m} \right) \cdot \beta_i \sigma_m$$

Which gives us the security market line

$$R_i = R_f + (R_m - R_f) \beta_i$$

V. The security market line says that, in equilibrium, the return on security $i$ is equal to the risk-free rate ($R_f$) plus the excess return on the market portfolio times the beta of security $i$. 