Case 1: Unsystematic risk only.

Recall that when the correlation $\rho$ between two securities equals zero, the portfolio variance is given by:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2$$

A simple generalization of this formula holds for many securities provided that $\rho = 0$ between all pairs of securities:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + \cdots + w_N^2\sigma_N^2. \quad (1)$$

We will prove the following result. As $N \to \infty$, the portfolio standard deviation $\sigma_p \to 0$.

To make the notation simpler, assume that $\sigma_1 = \sigma_2 = \cdots = \sigma_N = \sigma$. This means that each asset is equally risky. Under those circumstances, we try the simple diversification strategy of dividing our wealth equally among each asset such that $w_i = \frac{1}{N}$.

These assumptions allow us to rewrite expression (1) as

$$\sigma_p^2 = \left(\frac{1}{N}\right)^2\sigma^2 + \left(\frac{1}{N}\right)^2\sigma^2 + \cdots + \left(\frac{1}{N}\right)^2\sigma^2 \quad (2)$$

There are $N$ identical terms in expression (2), which means:

$$\sigma_p^2 = N\left(\frac{1}{N}\right)^2\sigma^2 = \frac{1}{N}\sigma^2$$

The expression in (3) shows that as $N$ grows larger and larger, the variance the portfolio declines. As $N \to \infty$, the variance goes to zero.

Case 2: Systematic and unsystematic risk

In fact, U.S. stocks do not have zero correlation with one another. We can capture the positive correlation of U.S. stocks with a factor model. Let $R_i$ denote the return on an
individual stock, and $R_M$ the return on a broad market index like the S&P 500. One way to capture the common source of variation is to run a regression of the values of $R_i$ on $R_M$:

$$R_i = \alpha_i + \beta_i R_M + \epsilon_i$$  \hspace{1cm} (3)

This is equivalent to fitting a line through a scatter plot of pairs of returns $(R_M, R_i)$. The slope of the line equals $\beta_i$. You may have heard in your statistics class that

$$\beta_i = \frac{\text{Cov}(R_i, R_M)}{\sigma_M^2}$$. \hspace{1cm} (4)

The coefficient $\beta_i$ is a measure of how much the stock moves together with the market index $R_M$. The error term, $\epsilon_i$, measures the variability in $R_i$ that is independent of all other securities in $R_M$.

Using (3), we can decompose the variance of a stock into its systematic and unsystematic components:

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_\epsilon^2$$  \hspace{1cm} (5)

Total risk = Systematic risk + Idiosyncratic risk

Here, $\sigma_M^2$ is the variance of $R_M$. Equation (5) follows from the fact that $\epsilon$ and $R_M$ are independent random variables.

Equation (5) shows that U.S. stocks have both systematic risk ($\beta_i^2 \sigma_M^2$) and unsystematic risk ($\sigma_\epsilon^2$). The argument from Case 1 demonstrates that the unsystematic component of stock risk goes away in a well-diversified portfolio (i.e. a portfolio with a large number of securities $N$). Only the systematic component remains.