Supply Chain Structure and Demand Risk

Ying-Ju Chen* and Sridhar Seshadri†

Stern School of Business, New York University

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Abstract

Agrawal & Seshadri [1] considered a problem in which a single risk neutral distributor supplies a short-lifecycle, long-leadtime product to several retailers that are identical except in their attitudes towards risk. They proved that the distributor should not offer the same terms to every retailer but instead offer less risky (from the demand risk perspective) contracts to more risk averse retailers. They did not prove the optimality of their menu.

In this paper we reconstruct their results when the number of retailers is infinite and their coefficient of risk aversion is drawn from a continuous distribution. We use optimal control theory to solve this problem. We show that this distribution uniquely determines the channel structure. Moreover, the optimal contract menu not only has the same structure as in Agrawal and Seshadri but is also optimal among nearly all contracts. The implications of these findings for channel design are discussed.

Keywords: Supply chain structure, Demand risk, Hidden information, Newsvendor model

1 Introduction

In the standard newsvendor model, a vendor offers a contract to price-taking retailers that face uncertain demand. The retailers choose an order quantity knowing that if the realized demand is larger than her quantity, excess demand will be met through an emergency purchase order at a higher price; otherwise, the unsold product will be re-sold at the salvage price. This contract will

*44 West 4th street, KMC 8-152, New York, NY 10012, USA; +1(212)998-0489, fax: 212-995-4003, e-mail: ychen0@stern.nyu.edu.
†Corresponding author; 44 West 4th Street, KMC 8-73, New York, NY 10012, USA; +1(212)998-0294, fax: 212-995-4003, email: sseshadr@stern.nyu.edu.
be called the “original newsvendor contract” (ONC). The ONC is common in many supply chains. Standard analysis shows that the optimal order quantity under the ONC is given by the “fractile rule” which depends on both the demand distribution as well as the retailer’s utility function.

Agrawal & Seshadri [1] showed that, if retailers have different risk preferences, the single contract offered by the vendor may not achieve the optimal risk reduction. Thus, in practice risk intermediation is often employed. A risk-neutral intermediary, also called the distributor in the sequel, can take the vendor’s ONC and instead offer a menu of contracts to the retailers. Since the distributor can absorb the risk at a lower cost (which is driven down to zero if the distributor is risk-neutral), the distributor is able to get benefits from offering risk-reducing contracts to retailers.\(^1\)

In Agrawal and Seshadri’s menu, less risky (from the demand risk perspective) contracts are given to more risk averse retailers. Such a menu of contracts increases the distributor’s expected profit because the retailers order more products. The intuitive content of this result is that the distributor can trade-off the expected value obtained by risk averse retailers against the gain in utility from risk reduction. They left unaddressed the question whether the menu of contracts designed by them is optimal.

In this paper we reconstruct their results when the number of retailers is infinite and their coefficient of risk aversion is drawn from a continuous distribution. We apply optimal control theory to solve the contract design problem. Surprisingly, the optimal menu not only has the same structure as that given by Agrawal and Seshadri but is also optimal among nearly all contracts. We also show that the distribution of the risk aversion coefficient uniquely determines the channel structure. Thus, distribution systems for products with long supply leadtimes and short lifecycles should bear marked similarities reflecting the attitude towards risk of channel participants.

The rest of the paper is organized as follows. In the next section we briefly discuss demand risk and its impact on ordering decisions, and then present the model in Section 3. Section 4 provides the optimal contract menu when all retailers are offered a contract, and in Section 5 we show that the proposed contract is optimal among all contract menus. Section 6 concludes with suggestions

\(^1\)The same logic applies in financial markets as well. For example, Allen & Gale [2] argue that speculators may manipulate the stock price if they are less risk-averse than uninformed traders.
for future work.

2 Demand risk

The impact of risk aversion on the order quantity has been examined in the framework of the “risk averse newsvendor problem.” In this problem, the retailer is offered the ordinary newsvendor contract (denoted as ONC) in which items that are ordered before the realization of demand are supplied at the cost of $c$ per unit, items ordered after demand has been realized at $e$ per unit, and unsold goods are taken back at $s$ per unit. For this problem, under the ONC, it is well known that the risk averse retailer’s order quantity (i.e., the one that maximizes his expected utility) will be smaller than the order quantity that maximizes his expected profit (see Baron [6], Eeckhoudt, Gollier, & Schlesinger [11], and Horowitz [18]). Obviously, the reduction in the order quantity of the retailer leads to lower expected profit (for the retailer). Eeckhoudt et al. give examples in which risk averse retailers will order nothing due to high demand uncertainty. Therefore, risk aversion of the retailers has been portrayed in the literature as leading to the loss of efficiency in supply chains. (We use the term efficiency to refer to the combined expected profit of the seller and the retailer. In general, this term refers to the total expected profit of all participants in a supply chain.)

Agrawal & Seshadri [1] showed not only that this loss of efficiency can be eliminated through risk reducing pricing contracts, but also that any risk neutral intermediary will find it beneficial to offer such risk reducing contracts to the retailers. In their model, the intermediary is referred to as the distributor\(^2\) who purchases the goods as per the terms of the ONC from the vendor, and in turn offers the goods to the retailers on contract terms that are less risky from the retailers’ viewpoint.

They proposed that, as opposed to the ONC, under the risk reducing contracts offered by the distributor to the retailers, the emergency purchase and the salvage prices should be set equal to the regular purchase price, and in addition a fixed payment should be made by the distributor to the retailer. Therefore, a retailer’s payoff consists of a fixed component (independent of the

\(^2\)The distributor can be an independent firm, or the vendor, or one of the retailers. For the sake of clarity we will refer to the intermediary as the distributor, and the risk averse players facing uncertain demand as the retailers. The analysis, though, is valid for any two levels in a vertical marketing channel, where the lower level facing uncertain demand is risk averse and the upper level is risk neutral (or less risk averse).
demand) and a variable component that increases linearly with the realized demand. Consequently, as the retailer’s payoff depends only upon the demand, the retailer is indifferent to the order quantity decision and is content to relegate the responsibility of determining an order quantity to the distributor. The distributor makes the order quantity decision fully aware that he has to satisfy all the demand faced by the retailer. The distributor bears the cost if necessary of buying the product at the emergency purchase cost and also the cost of disposing any unsold product at the salvage price.

By performing this function of “(demand) risk intermediation”, the distributor raises the retailers’ order quantities such that the maximum efficiency is obtained. The key contribution in their paper is to establish that the contracts offered to the retailer not only maximize the efficiency in the supply chain but are also optimal from the distributor’s viewpoint within the class of contracts that have a fixed payment and a linear price schedule. We show that such contracts are actually optimal for the distributor amongst a much broader class of contracts, thus making the menu designed by Agrawal and Seshadri much more attractive!

Contracts similar to the ones proposed by them are being adopted within the context of vendor managed inventory (VMI) programs. In many VMI programs the vendors make the inventory decisions on behalf of the retailers and also bear the risks and costs associated with these decisions (Andel [3]). In addition to the contracts found in VMI programs, we have observed several supply contracts, for example in the publishing, cosmetics, computers, apparel and grocery industries, that transfer the demand risk from the buyer to the vendor. In the publishing industry (Carvajal [8]) the retail outlets return their unsold magazines to the distributor at their purchase price and get additional shipments if they run out (such contracts were first introduced in the depression era). The terms of this contract are instrumental in persuading small retail merchants, who are averse to risk, to stock sufficient quantity of a wide variety of magazines.

Moses & Seshadri [29] describe the incentives used by the manufacturers in the cosmetics industry to increase the stocking level at department stores. The incentives in the cosmetic industry comprise of liberal return policies and the sharing of inventory holding costs between the producer and the department stores. In the personal computer industry (Kirkpatrick [21]), which is an industry plagued by steep price depreciation, the standard practice of vendors is to offer price
guarantees to their VAR (value added resellers) and hence the vendors absorb the risk of price erosion of their products during the period that they are held in the retailers’ inventory.

Similarly apparel retailers, who as an industry are facing increasing markdowns, are pressurizing their vendors to offer margin guarantees (Bird & Bounds [7]), i.e., the vendors are expected to absorb the markdown risk faced by the retailers in case the goods have to be disposed at “sale” prices. The grocery retailers’ response to increasing inventory risk due to the proliferation of SKU (Lucas [27]) is to charge a fixed slotting fees (similar to the fixed fee proposed by us), ranging from $5,000 to $20,000 per year per SKU, from the manufacturers/distributors irrespective of the sale volume of the items. The slotting fee is simply a rent charged for use of the shelf space. Therefore, we see a trend in the industry where simple price discounting contracts that were previously offered by the vendors (to induce the retailers to purchase a larger quantity) are being substituted by relatively sophisticated contracts that are designed to transfer the demand risk from the retailers to their vendors. Work on risk reducing contracts includes that of Chen, Sim, Simchi-Levi, & Sun [9], Donohue [10], Eppen & Iyer [12], Feng & Sethi [13], Fisher, Hammond, Obermeyer, & Raman [14], Fisher & Raman [15], and Gan, Sethi, & Yan [17]. In contrast to the majority of this work, which deals with a single retailer, we focus on the optimal contract for multiple risk averse retailers.

It is common knowledge how in the last two decades the concept of risk intermediation has been used to create not only novel investment and insurance products but also a global marketplace for such products and services. A large number of firms now offer a menu of products with different risk-return choices to customers worldwide. Viewed in this light, the existence of a similar market for hedging risky payoffs resulting from uncertain demand should not be entirely surprising. The contracts observed in some of the industries studied by us further confirm the insight provided by our analysis. It is also logical that such contracts are seen for products that have short life cycle or are perishable products such as grocery, personal computers and apparel, as these are the industries that are the most vulnerable to demand uncertainty. The use of the single period inventory model as the decision making framework embodied in the newsvendor problem is appropriate for such products as well.
3 Model

We consider a single period model in which multiple risk averse retailers purchase a single product from a common vendor. We assume that the retailers operate in identical and independent markets. The retailers face uncertain customer demand with a fixed selling price \( p \), and they accordingly make their purchase order quantity decisions to maximize their expected utility. The distribution of demand faced by a retailer is \( F_D(\cdot) \), which is independent of the contracts offered either by the vendor or by the distributor. The vendor has to offer the same supply contract(s) to each retailer. The terms of the contract offered to the retailers are to be determined.

Retailers are assumed to be risk averse, but have different degrees of risk aversion. We adopt a mean-variance utility approach, which can be regarded as the order-2 approximation of the original utility function via a Taylor series expansion. That is, let \( \rho_i \) denote the Arrow-Pratt risk aversion measure \( \rho_i \equiv -u''_i/u'_i \) where \( u_i(\cdot) \) is the retailer \( i \)'s original utility, and \( Z \) be a gamble. Then, retailer \( i \)'s expected utility is given by \( E[Z] - \rho_i \frac{\text{Var}[Z]}{2} \) (Pratt [30]). This approach is valid in the small gambles framework since higher-order terms vanish in the Taylor series expansion.

The decision problem of a retailer is to either select a contract from the menu offered by the distributor, or to accept the vendor’s ONC. An ONC is characterized by three per-unit parameters \( c, s, e \): \( c \) is the purchase price, \( s \) is the salvage value, and \( e \) is the emergency purchase price. We assume \( p \geq e \), thus all demand is met. In Agrawal and Seshadri’s model [1], the distributor offers a menu of contracts, each of which specifies only two terms: the fixed payment \( F(\rho) \), and the purchase/salvage/emergency price \( c(\rho) \). Later we will show that this restricted class of contracts is broad enough for constructing optimal contract menus.

Retailers are expected utility rather than expected profit maximizers. We define the reservation utility of retailer \( i \), denoted by \( r_i \), as the expected utility she will get upon accepting the vendor’s contract ONC. We assume that retailer \( i \) will choose a contract from the distributor’s menu if it provides at least an expected utility of \( r_i \). We show in Sec. 4 that \( r_i \)'s are ordered according to the coefficient of risk aversion.

In the main departure from Agrawal & Seshadri’s model [1], we assume that the coefficient of risk aversion \( \rho \) can take values in the interval \( [0, 1] \). We assume that it has the density function \( f_\rho(\rho) \). In this representation, the fraction of retailers in the population whose coefficient of risk
aversion lies in the interval $[\rho, \rho + dp]$ is given by $f_r(\rho)d\rho$. The distribution function of risk aversion and its complement are denoted by $F_r$ and $F_r^c$. We also assume that the reservation utility is a differentiable and convex function of $\rho$. Fig. 1 illustrates the supply chain structure.

![Figure 1: Supply chain contract structure.](image)

### 4 Main Results

In this section, we first review some results in Agrawal & Seshadri [1] where they consider a fixed number of retailers (instead of a continuum). The rest of the section focuses on the optimal contract menu with a continuum of retailers.

#### 4.1 Review of the model with discrete types

##### 4.1.1 Single contract

Agrawal and Seshadri find that the distributor has an incentive to cover fewer retailers if the distributor is allowed to offer only a single contract. The number of retailers covered decreases as the demand becomes more volatile (i.e., $\sigma/\mu$ increases), as the emergency cost $e$ increases, and, when the retailers’ margin $p - c$ increases. These results are a consequence of the fact that higher
s, e, or \( p - c \), allow the distributor to make greater profit per retailer. Thus with increasing s, e, or \( p - c \) it becomes more attractive for the distributor to “skim” the market and serve only the more profitable retailers.

4.1.2 Menu of Contracts

Let us recall the setting of Agrawal & Seshadri [1]. The set of all retailers is denoted by \( N \), where 
\[ N = \{1, \ldots, n\} \]
Retailer \( i \) has a reservation utility \( r_i \) and coefficient of risk aversion \( \rho_i \) where 
\[ \rho_i \leq \rho_j, \quad \forall i \leq j \]
Assume that the distributor offers a family of contracts \( C = \{F_i, c_i\} \), where the distributor makes payment \( F_i \) to retailer \( i \), supplies both regular and emergency orders at price \( c_i \) and also accepts returns at the same price \( c_i \). Denote the set of retailers that accept the contract as \( S(C) \). Note that the contract \( (F, p, p, p) \) is a risk-free contract under which the retailer gets a side payment \( F \) and passes the demand to the distributor at unit price of \( p \).

The following theorem summarizes their main results:

**Theorem 1.** (Agrawal & Seshadri [1]) In the optimal contract menu, there exists a fixed number \( k \) such that retailers \( k, k + 1, \ldots, N \) accept the risk free contract, \( F_k = r_k, c_k = p \), and the distributor’s profit is maximized by offering the contract \( (F_i, c_i) \), \( i \leq k - 1 \) given by

\[
c_i = p - \left( \frac{2(r_{i+1} - r_i)}{(\rho_{i+1} - \rho_i)\sigma^2} \right)^{0.5},
F_i + (p - c_i)\mu - \rho_i \frac{(p - c_i)^2\sigma^2}{2} = r_i.
\]

The distributor will offer a menu of contracts \( C^* = ((F_1, c_1), \ldots, (F_{k-1}, c_{k-1}), (r_k, p)) \), and every retailer will choose a contract, i.e., \( S(C^*) = N \). The expected value of contracts is ordered by \( \{r_i\} \)’s, which is decreasing in \( \rho_i \), and the distributor makes a profit on all contracts.

**Remark 1.** Agrawal & Seshadri [1] prove that the choice of \( ((F_1, c_1), \ldots, (F_{k-1}, c_{k-1}), (r_k, p)) \) eliminates the incentive of any retailer to select a contract that is not designed for her. In particular, as retailer \( k \) prefers the risk-free contract \( (r_k, p) \) to any other contracts, Property 5 in Agrawal & Seshadri [1] implies that all retailers \( j \geq k \) strictly prefer \( (r_k, p) \) to all others.

\[^3\text{In their paper, the ordering is reversed. We modify it here to make the discrete and continuous cases consistent.}\]
4.2 Optimal contract menu in the continuous case

We now discuss the model with a continuum of \(\rho \sim F_r(\cdot)\). We work with the probability triple \(([0,1], \mathcal{B}, F_r(\cdot))\) with \(\mathcal{B}\) being the Borel sets on \([0,1]\). We make the following assumption on the distribution of \(\rho\) in the sequel.

**Assumption 1.** \(xF_r^c(x)\) is unimodal and has a unique maximum at an interior point \(k \in (0,1)\).

**Remark 2.** In particular, Assumption 1 implies that the function \(xF_r^c(x) - xf_r(x)\) is initially positive and then becomes and stays negative. Note that if we interpret \(x\) as the price and the complementary cdf as the effective demand, \(xF_r^c(x)\) represents the revenue as a function of price. Its unimodality is commonly assumed in many papers on revenue management, e.g., Lariviere & Porteus [26] and Yoshida [34]. A sufficient condition for unimodality is when the distribution has the increasing generalized failure rate property (IGFR), namely, \(xf_r(x)/(1 - F_r(x))\) is increasing in \(x\). This is satisfied for the beta and the lognormal distributions (Lariviere [25]). Ziya, Ayhan, & Foley [35] compare three conditions that induce revenue unimodality, and mention some common distributions that satisfy these conditions such as normal, uniform, and gamma. The reader is referred to these papers and references therein for more details.

**Remark 3.** Note that this assumption is scale invariant, i.e., if \(x\) is scaled to \(bx\) then \(xF_r^c(x/b)\) remains unimodal. Moreover, the ‘point’ that achieves the maximum is also scale invariant. See the proof of Lemma 1 for details.

First, assume that all retailers are offered a contract and focus on the design of the optimal menu of contracts. Motivated by Agrawal & Seshadri [1], we formulate the optimal contract design problem in two stages. In the first stage, we assume that there exists a constant \(\tau \in [0,1]\) such that retailers with \(\rho \in [\tau,1]\) will choose the risk-free contract \((r(\tau), p, p, p)\) from the menu, where \(r(\rho)\) is the reservation utility of retailer with risk aversion coefficient \(\rho\). Note that \((r(\tau), p, p, p)\) is

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4 The support need not be \([0,1]\) for distributions we discuss here. We restrict to \([0,1]\) in this paper for merely notational ease. Another popular assumption for demand unimodality in revenue management is that the revenue is concave in demand, see Gallego & van Ryzin [16].

5 A simple algebraic computation shows that the triangle distribution is also unimodal. When the support is \([0,1]\), \(F_r^c(x) = 1 - 4x\), if \(0 \leq x \leq 1/2\) and \(F_r^c(x) = 3 + 4x\) if \(x \in [1/2, 1]\). Thus \(d(xF_r^c(x))/dx = 1 - 8x\) if \(x \in [0,1/2]\) and is \(3 + 8x\) while \(x \in [1/2, 1]\), which implies \(xF_r^c(x)\) is unimodal.
the cheapest risk-free contract that satisfies the participation constraints for all retailers \( \rho \in [\tau, 1] \) for two reasons: It is risk-free, so has the lowest expected value of all contracts that provide a utility of \( r(\tau) \). We show below that it provides utility greater than or equal to \( r(\rho) \) for all retailers with \( \rho \) in \([\tau, 1]\). We first develop optimal incentive compatible contracts to every retailer \( \rho \in [0, \tau) \) under such assumptions. In the second stage, we optimize over the choice of \( \tau \). Notice that we do not exclude the possibility of \( \tau = 1 \), i.e., only the most risk-averse retailer is offered the risk-free contract, and hence this is without loss of generality.

Consider any menu such that \( g(x) \) and \( h(x) \) are respectively the mean and variance of the payoff to a retailer if menu item \( x \) is chosen, where \( x \) can take values in the interval \([0, \tau]\) and \( \tau \in [0, 1]\). We therefore do not restrict ourselves to the family of contracts considered in Agrawal & Seshadri [1], and instead consider the most general form of the contracts. This is the most general form because retailers are concerned only about the mean and the variance of the payoff. With some abuse of notation, let a retailer with coefficient of risk aversion equal to \( x \leq \tau \) choose menu item \( x \). Given a fixed \( \tau \), the distributor’s problem is to choose \( f(g(x), h(x)) \) that solves the following maximization problem:

\[
\max \left\{ \left[ EV(S_{opt}^{EV}, c, s, e) - r(\tau) \right] F_D^c(\tau) + \int_0^\tau \left( EV(S_{opt}^{EV}, c, s, e) - g(\rho) \right) f_D(\rho) d\rho \right\},
\]

s.t. \( \rho \in \arg\max_{z \in [0, \tau]} g(z) - \rho h(z), \forall \rho \in [0, \tau) \),

\( r(\tau) \geq \max_{z \in [0, \tau]} g(z) - \rho h(z), \forall \rho \in [\tau, 1] \),

\( g(\rho) - \rho h(\rho) - r(\rho) \geq 0, \forall \rho \in [0, \tau) \),

\( r(\tau) \geq r(\rho), \forall \rho \in [\tau, 1] \),

where \( S_{opt}^{EV} = F_D(\frac{c}{\tau - s}) \) (\( F_D(\cdot) \) denotes the demand distribution) is the expected value maximizing order quantity as defined in Eq. (2) of Agrawal & Seshadri [1]. \( EV(S_{opt}^{EV}, c, s, e) \) is the expected cost that the distributor has to pay for buying the vendors’ ONC.

In Eq. (2), the first two inequalities are incentive compatibility (IC) conditions for respectively the retailers that receive a specific contract designed for her and the retailers that accept the risk-free contract. In (IC-1), we say that the contract menu is incentive compatible since the utility of retailer \( \rho \) is maximized if she chooses the contract with mean \( g(\rho) \) and variance \( h(\rho) \). On the
other hand, $r(\tau)$ is the utility of retailer $\rho \geq \tau$ when she receives the risk-free contract, and (IC-2) guarantees that she prefers this to any other contracts $g(z), h(z)$ with $z \in [0, \tau]$.

The last two inequalities in Eq. (2) represent individual rationality (IR) conditions, i.e., each retailer shall get at least her reservation utility. Note that the reservation utility $r(\rho)$ can be explicitly expressed as

$$r(\rho) = \max_S \{ E[\Pi(S, 0, c, s, e)] - \rho \frac{\text{Var}[\Pi(S, 0, c, s, e)]}{2} \},$$

where $\Pi(S, 0, c, s, e)$ is the profit if the ONC is accepted and the order quantity is $S$. Lemma 3.1 in Agrawal and Seshadri [1] shows that with this expression $r(\rho)$ is strictly decreasing in $\rho$, and hence the last inequality (IR-2) is automatically satisfied. To see this let $\rho_1 < \rho_2$. If retailer $\rho_1$ uses retailer $\rho_2$’s order quantity, she gets the same mean and variance but a higher expected utility because $\rho_1$ is less than $\rho_2$. If she optimizes the order quantity, then her expected utility can only be higher. Thus, $r(\rho_1) > r(\rho_2)$.

Our strategy is to first ignore the IC conditions for retailers $\rho \in (\tau, 1]$ (IC-2), and then verify that they are satisfied by our proposed menu. We assume that for retailer $\rho \in [0, \tau)$ the first order condition for interior optimality (or local optimality – LO) hold:

$$\left(LO\right) \left[ \frac{dg(z)}{dz} - \rho \frac{dh(z)}{dz} \right]_{z=\rho} = 0, \ \forall \rho \in [0, \tau]. \quad (3)$$

We shall replace constraints (IC-1) and (IC-2) in Eq. (2) by (LO), and obtain the necessary conditions for optimality for the modified version of the problem. A candidate menu will then be proposed based on this relaxed optimization problem, and later we prove that (LO) for that menu ensures that each retailer $\rho \in [0, \tau)$ is choosing her contract optimally.

Denote the expected value of the utility obtained by retailer $\rho$ as $\hat{r}(\rho) = g(\rho) - \rho h(\rho)$. Using Eq. (3) gives

$$\frac{d\hat{r}(\rho)}{d\rho} = \left[ \frac{dg(\rho)}{d\rho} - \rho \frac{dh(\rho)}{d\rho} \right] = -h(\rho). \quad (4)$$

This implies that

$$g(\rho) = \hat{r}(\rho) + \rho h(\rho) = \hat{r}(\rho) - \rho \frac{d\hat{r}(\rho)}{d\rho}. \quad (5)$$

Now we come back to the distributor’s optimization problem Eq. (2). Observing that the term $[EV(S_{opt}^{EV}, c, s, e) - r(\tau)]F_r^c(\tau)$ is independent of the choice of $\{(g(\rho), h(\rho)), \ \rho \in [0, \tau]\}$, the
The distributor’s problem becomes:

\[ \max \int_0^\tau \left[ -\dot{r}(\rho) + \rho \frac{d\dot{r}(\rho)}{d\rho} \right] f_r(\rho) d\rho \]  \hspace{1cm} (6)

subject to

\[ \dot{r}(\rho) \geq r(\rho), \quad \dot{r}(\tau) = r(\tau), \quad \dot{r}(0) = r(0). \]  \hspace{1cm} (7)

(Why pay the risk neutral retailer more than the expected value? Therefore \( \dot{r}(0) = r(0) \).) We can rewrite the problem as

\[ \max \int_0^\tau \left[ -\dot{r}(\rho) + r(\rho) + \rho \frac{d\dot{r}(\rho)}{d\rho} - \rho \frac{dr(\rho)}{d\rho} \right] f_r(\rho) d\rho, \]  \hspace{1cm} (8)

subject to Eq. (7) since the added terms are independent of the policy \( g(\rho) \). Let \( x(\rho) \equiv \dot{r}(\rho) - r(\rho) \) be the state variable, and \( u(\rho) = d(\dot{r}(\rho) - r(\rho))/d\rho \) be the control. Through this transformation, the design of the optimal menu of contracts can be recast as an optimal control problem and can be solved by use of calculus of variation.\(^6\) The Hamiltonian is given by

\[ H(\rho) = (-x(\rho) + \rho u(\rho)) f_r(\rho) + \lambda(\rho) u(\rho). \]  \hspace{1cm} (9)

The adjoint equation is given by

\[ \frac{d\lambda(\rho)}{d\rho} = -\frac{\partial H}{\partial x} = f_r(\rho), \]  \hspace{1cm} (10)

and the transversality condition gives no information. Let \( \lambda(\tau) = c \) where \( c \) is some constant, we obtain

\[ \lambda(\rho) = c - F^\tau_r(\rho). \]  \hspace{1cm} (11)

The necessary condition for optimality is that the Hamiltonian is maximized by the choice of \( u \).

From Eqs. (9) and (11), \( H \) is linear in \( u \), and the coefficient of \( u \) in \( H \) is

\[ c + \rho f_r(\rho) - F^\tau_r(\rho). \]  \hspace{1cm} (12)

\(^6\)A similar approach can be found in Laffont & Tirole [24].
If the expression in Eq. (12) were positive, the solution would be unbounded. Due to our assumption about the uniqueness of the maximum, it is not hard to see that $c = 0$. Note that $\rho f_r(\rho) - F_r^c(\rho)$ is the derivative of $-\rho F_r^c(\rho)$, and hence from Assumption 1, $\rho f_r(\rho) - F_r^c(\rho) > 0$ if $\rho > k$, and $\rho f_r(\rho) - F_r^c(\rho) < 0$ if $\rho < k$. The case $\rho = k$ has measure zero and hence it will not contribute to the integral Eq. (8). If $\rho f_r(\rho) - F_r^c(\rho) > 0$, there is no maximum since we can take $u \to \infty$. On the other hand, when $\rho f_r(\rho) - F_r^c(\rho) < 0$ we should make $u$ as negative as possible. But, the boundary conditions $\hat{r}(\rho) \geq r(\rho)$ on $[0, \tau]$ on the other hand they require that $u(\rho)$ be greater than or equal to zero whenever $\hat{r}(\rho) = r(\rho)$. It therefore follows that $\hat{r}(\rho) = r(\rho)$ for all $\rho$ in $[0, \tau]$.

The following theorem summarizes our results thus far.

**Theorem 2.** Suppose Assumption 1 holds, retailers with $\rho \in [\tau, 1]$ choose contract $(r(\tau), p, p, p)$, where constant $\tau \in [0, 1]$, and the distributor has to serve all retailers. Then the necessary conditions for the optimal contract menu are (i) (LO) in Eq. (3) and (ii) retailers $\rho \in [0, \tau]$ receive their reservation utilities.

### 4.2.1 Candidate menu and verifying the necessary and sufficient conditions

Now we will propose a candidate menu of contracts. The inspiration is due to the optimal menu in the discrete version, i.e., the one proposed in Theorem 1. We will focus on the class of contracts with a fixed franchise fee and common cost $\{F(\rho), c(\rho)\}$, and prove that this class is broad enough to achieve the optimality. Passing to the limit in Eq. (1), the cost $c(\rho)$ charged to the retailer with a coefficient of risk aversion equal to $\rho$ and the corresponding fixed side payment $F(\rho)$ are given by the solution to

$$c(\rho) = p - \left(-\frac{2d r(\rho)}{d \rho} \frac{1}{\sigma^2}\right)^{0.5},$$

$$F(\rho) + (p - c(\rho))\mu - \rho \left(\frac{(p - c(\rho))^2}{2}\sigma^2\right)^{0.5} = r(\rho).$$

The corresponding $g(\rho)$ and $h(\rho)$ are $F(\rho) + (p - c(\rho))\mu$ and $\left(\frac{(p - c(\rho))^2}{2}\sigma^2\right)^{0.5}$, $\forall \rho \in [0, \tau]$.

We will verify now that the proposed contract menu satisfies the necessary and sufficient conditions.
Checking condition (ii) in Theorem 2.

With the menu shown in Eq. (13), retailers with \( \rho \in [0, \tau] \) receive their reservation utilities.

Checking (IC-1).

We now verify that (LO) implies global optimality for \( \rho \in [0, \tau] \). Suppose a retailer \( \rho \in [0, \tau] \) chooses the contract designed for retailer \( z \in [0, \tau] \). The fixed side payment is \( F(z) = r(z) - (p - c(z))\mu + z\frac{2(p - c(z))\sigma^2}{2} \), and hence her payoff by doing so will be

\[
F(z) + (p - c(z))\mu - \rho \frac{(p - c(z))^2\sigma^2}{2} = r(z) - (\rho - z)\frac{(p - c(z))^2\sigma^2}{2}.
\]

From Eq. (13), we have \( p - c(z) = \left( -\frac{2dr(y)}{dy} \right)_{y=\frac{1}{\sigma^2}} |_{y=z}^{0.5} \), and hence retailer \( \rho \)'s payoff becomes \( r(z) - (\rho - z)\frac{dr(y)}{dy} |_{y=z} \). Recall that if she chooses her own contract, she receives her reservation utility \( r(\rho) \). Thus, (IC-1) boils down to

\[
r(\rho) \geq r(z) + (\rho - z)\frac{dr(y)}{dy} |_{y=z}, \quad \forall z \in [0, \tau],
\]

which is equivalent to saying that \( r(\rho) \) is convex.

The convexity of \( r(\rho) \) is established in Lemma 3.2 of Agrawal & Seshadri [1], and we briefly present the proof here. Suppose that \( \rho_1 < \rho_2 < \rho_3 \) and \( S_2 \) is the optimal ordering quantity under the ONC for retailer with \( \rho_2 \). If retailers with \( \rho_1 \) and \( \rho_2 \) use \( S_2 \) as the ordering quantity, we have

\[
r(\rho_2) = g(S_2) - \rho_2 h(S_2) \quad \text{and} \quad r(\rho_1) \geq g(S_2) - \rho_1 h(S_2) \quad \text{Thus} \quad r(\rho_1) - r(\rho_2) \geq (\rho_2 - \rho_1) h(S_2),
\]

which yields \( h(S_2) \leq \frac{r(\rho_1) - r(\rho_2)}{\rho_2 - \rho_1} \). On the other hand, if using \( S_2 \) as the ordering quantity for both retailers with \( \rho_2 \) and \( \rho_3 \), we obtain

\[
r(\rho_2) - r(\rho_3) \leq (\rho_3 - \rho_2) h(S_2) \quad \Rightarrow \quad h(S_2) \geq \frac{r(\rho_2) - r(\rho_3)}{\rho_3 - \rho_2}.
\]

Combining both cases, we have \( \forall \rho_1 < \rho_2 < \rho_3 \),

\[
\frac{r(\rho_1) - r(\rho_2)}{\rho_2 - \rho_1} \geq \frac{r(\rho_2) - r(\rho_3)}{\rho_3 - \rho_2} \iff \frac{r(\rho_2) - r(\rho_1)}{\rho_2 - \rho_1} \leq \frac{r(\rho_3) - r(\rho_2)}{\rho_3 - \rho_2}.
\]

Therefore \( r(\rho) \) is convex in \( \rho \).

Since \( r(\rho) \) is convex, Eq. (14) is valid, and retailer \( \rho \)'s payoff attains its maximum when contract \((F(\rho), c(\rho))\) is selected, \( \forall \rho \in [0, \tau] \). Thus, (IC-1) is true.
\textbf{Checking (IC - 2).}

The IC condition for retailer $\tau$ yields $r(\tau) \geq F(z) - (p - c(z))\mu - \tau \frac{\sigma^2(p - c(z))^2}{2}$, $\forall z \in [0, \tau)$, and hence for retailer $\rho \in (\tau, 1]$, choosing the risk-free contract (which gives $r(\tau)$) is strictly preferred since

$$r(\tau) \geq F(z) - (p - c(z))\mu - \tau \frac{\sigma^2(p - c(z))^2}{2} > F(z) - (p - c(z))\mu - \rho \frac{\sigma^2(p - c(z))^2}{2}, \ \forall z \in [0, \tau).$$

The above discussions establish the necessity of optimality given a fixed $\tau$.

\textbf{Checking the sufficiency.}

As the Hamiltonian Eq. (9) is linear in the state variable, the derived Hamiltonian is concave in the state variable and satisfies the sufficient condition for optimality (see Theorem 2.2 of Sethi & Thompson [33]).

\textbf{Remark 4.} The way we show the optimality here is different from the standard approach used in the literature of nonlinear pricing. The standard approach (e.g., see Salanie [32]) is to first assume the single-crossing (sorting) condition, or so-called Spence-Mirrlees condition, whose definition is given below. Suppose that a retailer’s payoff is $F - u(q, \rho)$, where $F$ is the monetary transfer, $q$ is the quality level (contract terms), and $\rho$ is the retailer’s unobservable “type” (the coefficient of risk aversion). The single-crossing condition, labelled as (SC), requires that $\frac{\partial^2 u(q, \rho)}{\partial q \partial \rho} < 0$, $\forall q$. In words, this condition ensures that types can be ranked according to their marginal utilities, and it implies that utilities of two distinct retailers intersect at most once. This condition is commonly adopted in the literature of nonlinear pricing, Nash implementation (Maskin [28]), and auctions (Krishna [22]).

With the single-crossing condition, it can be shown that the necessary and sufficient conditions for (IC-1) and (IR-1) conditions are (LO) and the monotonicity (M) of $h(\rho)$, see Salanie [32] for details. We now verify that for our proposed menu, both (SC) and (M) hold.

\textbf{Checking (M).}

The term corresponding to the variance $h(\rho)$ is $\frac{\sigma^2(p - c(\rho))^2}{2} = -\frac{\partial r(y)}{\partial y} |_{y=\rho}$, which is indeed monotonic in $\rho$ from the convexity of $r(\rho)$.

\textbf{Checking (SC).}

Define $q(\rho) = \frac{\sigma^2(p - c(\rho))^2}{2}$. Recall that the retailers possess mean-variance utility $g(z) - \rho h(z)$,
and hence the mean \( (g(z)) \) does not contribute to \( \frac{\partial^2 u(q, \rho)}{\partial \rho \partial q} \). Thus \( \frac{\partial^2 u(q, \rho)}{\partial \rho \partial q} = -1 < 0, \forall q \geq 0, \) i.e., (SC) is satisfied.

### 4.2.2 Optimal choice of \( \tau \)

Now we turn to the second stage: optimizing over the choice of \( \tau \). Let \( \Xi(\tau) \) denote the profit function of the distributor when retailers that have a coefficient of risk aversion greater than \( \tau \) are offered the risk-free contract. Using Eq. (13), \( \Xi(\tau) \) can be restated as

\[
\Xi(\tau) = (EV(S_{\text{opt}}^{EV}, c, s, e) - r(\tau))F_{\tau}^c(\tau) + \int_{0}^{\tau} (EV(S_{\text{opt}}^{EV}, c, s, e) - r(\rho) + r \frac{dr(\rho)}{d\rho}) f_r(\rho) d\rho. \tag{15}
\]

Using the rule for differentiating under the integral we obtain

\[
\frac{d\Xi(\tau)}{d\tau} = -\frac{dr(\tau)}{d\tau} (F_{\tau}^c(\tau) - \tau f_r(\tau)). \tag{16}
\]

From Eq. (16) and the fact that \( -\frac{dr(\tau)}{d\tau} > 0 \), the maxima of the profit function are independent of the reservation utility. Moreover, note that the expression in parentheses in Eq. (16) is the derivative of \( \tau F_{\tau}^c(\tau) \). Therefore, if the function \( \tau F_{\tau}^c(\tau) \) has a unique maximum in the interior of \([0, 1]\), then the optimal value of \( \tau \) is independent of the reservation utility. In other words, the fraction of retailers who select the risk free contract is independent of product characteristics if the distribution is unimodal.

Recall that \( k \in (0, 1) \) is the value of \( \tau \) at which the function \( \tau F_{\tau}^c(\tau) \) attains its maximum. Thus,

\[
F_{\tau}^c(\tau) - \tau f_r(\tau) \geq 0, \ \tau \in [0, k], \tag{17}
\]

and the necessary condition for optimality of \( \Xi(\tau) \) is \( \tau = k \). We use \( C^* = \{F^*(\rho), c^*(\rho)\} \) to denote the contract menu where \( (F^*(\rho), c^*(\rho)) \) are as given in Eq. (13) and the corresponding payment when \( \rho \in [0, k) \) and \( (F^*(\rho), c^*(\rho)) = (r(k), p), \forall \rho \in [k, 1] \). Notice that in the continuous case the menu \( C^* \) we propose again gives a risk-free contract to all retailers with coefficient higher than \( k \), which is chosen in Eq. (16). This completes the characterization of the optimal menu of contracts, and therefore we have
Theorem 3. Suppose Assumption 1 holds. Let \( k = \arg \max_{\tau \in [0,1]} \tau F^c_r(\tau) \) and \( C^* = \{(F^*(\rho), c^*(\rho)), \rho \in [0,k]\} \). Then the proposed \( C^* \) is optimal among the class of menus that serve all retailers. Moreover, under the optimal menu of contracts, all retailers \( \rho \in [0,k] \) receive their reservation utilities, and retailers \( \rho \in (k,1] \) are offered the same risk-free contract.

Note that the class of menus we consider include all menus since retailers’ utility functions are of the mean-variance format. Hence, if all retailers ought to be served, \( C^* \) is indeed the optimal menu.

Remark 5. If we assume instead the distributor can offer a contract to only one retailer, it becomes an adverse selection problem. This may be of interest to study in future work.

5 Verification of the optimality

In Theorem 3, we have shown that if all retailers are served, our proposed contract menu \( C^* \) yields the highest expected payoff to the distributor. The purpose of this section is to show that our proposed menu of contracts is indeed optimal even when we allow the distributor to exclude some retailers (for example, offer contracts only to those whose coefficient of risk aversion falls in \([0,0.25) \cup [0.7,0.993]\)). We do this through three lemmas and a theorem as stated below. The proofs are given in the appendix.

Let \( S(C) \) be the set of retailers that receive and accept contracts from the menu \( C \). For each \( x \in S(C) \), the menu \( C \) specifies a bundle \((F(x), c(x))\). Needless to say, the sets \( S(C) \) of interest should be measurable with respect to the probability space \(([0,1], \mathcal{B}, F_r(\cdot))\). Due to the special structure of our proposed contract, we show that if the distributor wants to serve merely the retailers on an interval \( I \subset [0,1] \) and ignore all other retailers, the optimal one-segment contract menu coincides with the proposed contract \( C^* \) restricted to the interval \( I \) (denoted as \( C^*|_I \)):

Lemma 1. (Decomposition) Suppose \( C^* = (F^*(x), c^*(x)) \) is the optimal contract menu for \( S(C^*) = [0,1] \). Then for any interval \( I \subset [0,1] \), \( C^*|_I \) is also optimal.

This lemma says for any given contract \( C \) with arbitrary number of segments, the distributor will be better off if she replaces \( C \) by menu \( C^* \) in every segment. Next we will study two properties of the proposed contract menu \( C^* \), namely the no-skip property and push-to-the-end property.
Lemma 2. (No-skip property) Suppose the distributor adopts menu $C^*$ and $S(C^*)$ is composed of two disjoint intervals $I_1$ and $I_2$, then the distributor will be better off by offering contracts to all retailers in $I_1$, $I_2$, and also those between $I_1$ and $I_2$.

Applying this lemma inductively, we obtain that if the distributor offers the menu $C^*$, then the optimal $S(C^*)$ will be an interval. The following lemma says that while offering family of contracts $C^*$, the distributor should not leave any uncovered intervals of retailers from both ends.

Lemma 3. (Push-to-the-end property) Suppose the distributor adopts menu $C^*$ and $S(C^*) \neq \emptyset$. Let $\bar{s} \equiv \sup \{x : x \in S(C^*)\}$ Then it is in the distributor’s interest to set $\bar{s} = 1$. On the other hand, if $\underline{s} \equiv \inf \{x : x \in S(C^*)\}$, the distributor will set $\underline{s} = 0$.

Combining Lemmas 1-3, if the distributor offers $C^* = \{F^*(x), c^*(x)\}$, she will offer contracts to the entire interval $[0, 1]$ to maximize her profit. Bearing in mind the structure of $C^*$, we are ready to prove its optimality among all feasible contract menus:

Theorem 4. The proposed contracts $(F^*(x), c^*(x))$ are optimal among all contracts that offer a menu to a measurable set of retailers.

In the literature of nonlinear pricing, the “second-best” scenario refers to the case where the person offering a menu does not know private information of those players she is proposing the contracts to, which coincides with our model as the distributor is unable to observe the risk aversion levels of retailers. In our model, the second-best contract menu $C^*$ enables the distributor to extract all the information rent of retailers who are less risk averse, while leaving the retailers with higher risk aversion the full information rent. This result is in strict contrast to the standard case where players are endowed with the same reservation utilities. In our model the reservation utility of a retailer comes from her alternative “accepting the ONC.” Therefore, the reservation utility varies from type to type in nature, and is decreasing in $\rho$. To induce the retailers with lower $\rho$ to participate, the distributor should give them higher values of utility, thus the IC constraints and IR conditions of retailers with low $\rho$ are both binding. This corresponds to Case 2 of Sec. 3.3.1 in Laffont & Martimort [23] where the discrete case is discussed.

The fact that a continuum of retailers receive a risk-free contract is also worth noting. It is known as the “bunching” phenomenon (Laffont & Martimort [23]), which may occur in the standard
case when the monotone hazard rate property of types fails. Here the bunching occurs in retailers with high risk aversion and the contract offers them the efficient level, i.e., it fully covers the demand risk for those risk averse retailers. More discussion on type-dependent participation constraints can be found in Jullien [20].

Finally, we show in the following corollary that the proposed menu $C^*$ is unique up to a measure-zero modification, which means all menus properly different from $C^*$ are suboptimal.

**Corollary 1.** The menu $C^*$ is the cheapest menu that achieves the optimal profit uniquely up to a measure-zero set.

### 6 Conclusion and Extension

In this paper we show that the contract menu proposed by Agrawal & Seshadri [1] is indeed optimal among all possible menus, provided that the distribution of risk aversion is continuous and satisfies some mild condition commonly adopted in the revenue management literature. The channel structure is uniquely determined by the distribution, independent of the underlying ONC contract and the demand distribution.

The same results hold for other cases if there exists a parameter $y$ and two functions $g(\cdot), h(\cdot)$ such that the utility of a type-$y$ retailer receiving the contract $C$ is $g(C) + yh(C)$, and the reservation utility is differentiable and decreasingly convex in $y$ (required in Eqs. (13) and (16)). If the payoff of retailers is normally distributed, such an utility structure may show up since the first and second moments are sufficient statistics for all of its moments.

The reason why it does not work for a general utility function (e.g., $g(C) + yh(C) + \delta(y, C)$ where $\delta(y, C)$ is the higher order term) can be seen by examining the proof of Theorem 3. With this extra term $\delta(y, C)$, Eqs. (4) and (5) both fail, and therefore the optimal control problem cannot be solved simply by use of calculus of variation. Further investigation on general utility functions is needed, especially when the retailers cannot be ordered by a single parameter.$^7$

$^7$Similarly, a significant complication occurs in nonlinear pricing while introducing “multi-dimensional” types, see Armstrong [4] and Jehiel, Moldovanu, & Stacchetti [19].
A Proofs in Sec. 5.

A.1 Proof of Lemma 1

Let \( k \) be the cutoff point such that under contracts \( C^* \) retailers in \([0, k]\) get the risk-free contract. First we consider the case when \( I \subset [k, 1] \). In this case, if the distributor provides \( C^*_I \) to the retailers, their IC conditions are satisfied and their rents are fully extracted. Thus, no other contract will yield a higher expected revenue for the distributor. If \( I \nsubseteq [0, k] \), we can derive the optimal one-segment contract \( C' \) on \( I \). For ease of explanation we define \( I = [\alpha, \beta] \). Recall that the retailers are located on \( I \) with a rescaled distribution \( F'_r \) where

\[
F'_r(x) = \begin{cases} 
0, & x < \alpha, \\
\frac{F_r(x) - F_r(\alpha)}{F_r(\beta) - F_r(\alpha)}, & \alpha \leq x \leq \beta, \\
1, & x \geq \beta.
\end{cases}
\] (18)

Now we revisit the second equation of the first-order condition, we find that

\[
(F'_r)'(x) - x f'_r(x) = \frac{1}{F_r(\beta) - F_r(\alpha)} [F_r(x) - x f_r(x)]
\] (19)

has again \( k \) as the cutoff point since \( \frac{1}{F_r(\beta) - F_r(\alpha)} \) is an irrelevant constant. Thus, the new contract \( C' \) on \( I \) offers the risk-free contract to retailers in \([\alpha, k]\) and extracts retailers’ rent in \([k, \beta]\), which exactly coincides with \( C^*_I \).

A.2 Proof of Lemma 2

It suffices to consider the case with closed intervals since the probability measure \( F_r \) is atomless. Suppose that these two intervals are \([\alpha_1, \beta_1]\) and \([\alpha_2, \beta_2]\) with \( \beta_1 < \alpha_2 \). Since these two intervals are disjoint, we can find two points \( a, b \) such that \( \beta_1 < a < b < \alpha_2 \). Let \( C' = C'' \cup C^* \) where \( S(C'') = (a, b) \) and \( C'' = \{(F^*(x), c^*(x)) \; \forall x \in S(C'') \} \), i.e., we propose contracts \((F^*(x), c^*(x))\) to those retailers on \((a, b)\). Since the choice of \((F^*(x), c^*(x))\) directly ensures their IC conditions, these retailers on \([\alpha_1, \beta_1]\) and \([\alpha_2, \beta_2]\) will not deviate to choose any contract of \( C'' \). The IC conditions for a retailer \( x \) on \((a, b)\) again follow from the construction of \((F^*(\cdot), c^*(\cdot))\). By Theorem 3, retailer \( x \) in \([a, b]\) receives her reservation utility \( r(x) \).
We now state and use Lemma 2.2 in Agrawal & Seshadri [1]. Suppose \( F_x \) and \( F_y \) are distribution functions of respectively random variables \( X \) and \( Y \), and \( F^{-1}(a) = \inf\{ b \in R : F(b) \geq a \} \) denotes the “inverse” of distribution \( F \). \( X \) is said to be less than \( Y \) in the dispersive order if and only if

\[
F_x^{-1}(\omega_2) - F_x^{-1}(\omega_1) \leq F_y^{-1}(\omega_2) - F_y^{-1}(\omega_1), \quad \forall 0 < \omega_1 \leq \omega_2 < 1.
\]

Lemma 2.2 of Agrawal & Seshadri [1] says that

\[
\Pi(S_{opt}^{EV}, F^*(x), c^*(x), c^*(x)) = F^*(x) + (p - c^*(x))D,
\]

where \( D \) denotes the realized demand, is smaller than the profit under the ONC in the dispersive order. Therefore, the distributor gets positive payoff from offering the contract \((F^*(x), c^*(x))\) to retailer \( x \), which completes the proof.

A.3 Proof of Lemma 3

Suppose \( \bar{s} < 1 \), then there exists an interval \((a, b)\) such that \( \bar{s} < a < b < 1 \). Let \( C'' = C \cup C' \), where \( S(C') = (a, b) \) and \( C' = \{(F^*(x), c^*(x)), \forall x \in (a, b)\} \). Since the IC conditions of any \( x \in S(C'') \) are satisfied, no deviation can occur. Hence the distributor receives a higher payoff under \( C'' \) than under \( C^* \), which contradicts the assumption that \( C^* \) is optimal. Therefore setting \( \bar{s} = 1 \) is in the distributor’s interest.

On the other hand, let us suppose that \( \bar{s} > 0 \). Lemma 3.5 in Agrawal & Seshadri [1] implies that when \( \rho \) is discrete, if \( \bar{s} > 0 \), then the distributor will find it profitable by offering a contract to a retailer with \( \rho < \bar{s} \). A similar argument shows that in the continuous case, the distributor will be better off by offering contracts to retailers in \((c, d)\), where \( 0 < c < d < \bar{s} \). Thus \( \bar{s} = 0 \) in the optimal menu.

A.4 Proof of Theorem 4

Let \( \hat{C} \) be the family of contracts that is optimal and \( S(\hat{C}) \) be its associated set. Since \( S(\hat{C}) \) has to be measurable and the probability measure of \( \rho \) is equivalent to the Lebesgue measure, for an arbitrarily small constant \( \epsilon \), we can find a closed set \( G \subset S(\hat{C}) \) such that the measure of \( \{ \rho \in G \} \) is greater than or equal to the measure of \( \{ \rho \in S(\hat{C}) \} - \epsilon \). Moreover, \( G \) is compact by its closedness and the fact \( G \subset [0, 1] \), and hence there exists a finite open covering \( O \) that covers set \( G \) (see, e.g., Royden [31]). Grouping all the connected sets, we can decompose \( O \) into a finite number of disjoint
open components, i.e., \( O = \sum_{j=1}^{J} O_j \) where \( \{O_j\} \)'s are mutually non-overlapping open intervals and \( J \) is the number of these intervals.

Define \( \pi(C) \) as the expected payoff if the distributor adopts menu \( C \). Now we will propose a new menu of contracts \( C' \) whose set \( S(C') \) is an interval and \( \pi(C') \geq \pi(\tilde{C}) - M\epsilon \), where \( M \) is the maximal payoff that the distributor can gain by offering a contract to a retailer. \( M \) should be bounded (otherwise the distributor extracts infinite profit from a single retailer!) and can be found by solving the single-retailer problem:

\[
M \leq \sup_{\rho \in [0,1]} \max_{(F,c')} \{E[\Pi(S_{opt}^{EV}(0,c,s,e)) - (F + (p - c')\mu) : F + (p - c')\mu - \rho(p - c')2^{2}\geq r(\rho)]\},
\]

which is bounded by the continuity of the objective and the compactness of \([0,1]\). Note also that \( M \) is a fixed constant independent of the choice of \( \tilde{C} \).

First, we replace the contracts offered to those retailers on \( O \) by \( C'' \equiv (F^*(\rho),c^*(\rho)) \) and \( S(C'') = O \). On each open interval \( O_j \), Lemma 1 shows that it is optimal within this interval \( O_j \) and hence the payoff that the distributor gets from retailers in \( O_j \) under \( C'' \) is higher than that under \( \tilde{C} \). Moreover, since the IC conditions under \( C'' \) are a subset of the IC conditions under \((F^*(\rho),c^*(\rho))\) on \([0,1]\), a retailer \( \rho \) will weakly prefer to choose her own \((F^*(\rho),c^*(\rho))\) over all other contracts \((F^*(\rho'),c^*(\rho'))\), \( \forall \rho' \in O \). Note that there are some retailers in \( O \) that are not considered in \( \tilde{C} \) before. Property 8 in Agrawal & Seshadri [1] says that if a retailer receives the same utility while accepting a contract as that under the ONC, then the distributor will earn positive profit from this retailer. Thus, offering contracts to these retailers while keeping others’ contracts fixed still satisfies all IC conditions and can only benefit the distributor. Thus from the retailers on \( O \), the distributor gets at least as much payoff under menu \( C'' \) as what she gets on \( G \) under \( \tilde{C} \).

If \( J = 1 \), i.e., the open subcovering \( O \) is itself an interval, we can define \( C' = C'' \), and the distributor’s payoff under \( C' \) is \( \pi(C') \geq \pi(\tilde{C}|G) \geq \pi(\tilde{C}) - M\epsilon \), where \( \pi(\tilde{C}|G) \) denotes the expected payoff that the distributor gets from retailers in \( G \in S(C) \) by offering contract menu \( C \). While \( G = S(C) \) we suppress the notation to \( \pi(C) \).

Next we focus on the case \( J > 1 \). Let \( \alpha_i, \beta_i \) be respectively the left-hand and right-hand endpoints of \( O_j \), and we assume without loss of generality that \( \beta_i < \alpha_{i+1}, \forall i \in \{1,...,J-1\} \). Since \( O_j \) and \( O_{j+1} \) are disjoint where \( j \leq J - 1 \), we can find an open interval \((a,b)\) such that
\[ \beta_j < a < b < \alpha_{j+1}. \] Note that \((a, b)\) has strictly positive measure. Let \(C' = C'' \cup C^*\) where \(C^* = (F^*(\rho), c^*(\rho))\) and \(S(C^*) = (a, b)\), i.e., we propose contracts \((F^*(\rho), c^*(\rho))\) to those retailers on \((a, b)\). Applying Lemma 2, we know that all these retailers on \(O\) will not deviate to choose any contract of \(C^*\). The IC conditions on \((a, b)\) are again satisfied immediately from the construction of \(C^*\).

Consequently, as long as \(O\) is not connected, we can always offer contracts to some retailers in between two intervals \(O_j\) and \(O_{j+1}\) and yield a (weakly) higher payoff. We then obtain by induction that the optimal contract \(C'\) with \(O \subset S(C')\) has the “no-skip” property (Lemma 2), i.e., \(S(C')\) is an interval. The distributor’s payoff under \(C'\) is \(\pi(C') \geq \pi(C'|O) \geq \pi(\bar{C}|G) \geq \pi(\tilde{C}) - M\epsilon\), where the second inequality follows from Lemma 1.

So far we have established that for a given \(\epsilon\), there exists a contract menu \(C'\) such that \(\pi(C') > \pi(\bar{C}) - M\epsilon\) and \(S(C')\) is an interval. Finally, since \(\epsilon\) can be arbitrarily small and \(M\) is fixed, the distributor’s payoff under our proposed contracts can be made arbitrarily close to the optimal level, which completes the proof.

A.5 Proof of Corollary 1

An argument similar to the proof of Lemma 1 shows that in every interval, our contract menu \(C^*\) is the cheapest menu that extracts the reservation utility \(r(\rho)\) from retailers with \(\rho \in [k, 1]\). If there exists another contract \(\hat{C}\) such that \(\hat{C}\) extracts retailers’ reservation as well and is cheaper than \(C^*\), then it can be cheaper in at most a set of countable points (otherwise we would have found an interval over which \(\hat{C}\) outperforms \(C^*\)). Since the distribution \(F_\tau\) is equivalent to Lebesgue measure (it has a density), every single point has measure zero, and a countable union of measure zero points also has measure zero (Ash & Doleans-Dade [5]). Thus, \(C^*\) is uniquely optimal up to a measure-zero set.
References


