Dynamic Pricing for Vertically Differentiated Products

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Abstract

This paper studies the seller’s optimal pricing policies for a family of substitute perishable products. The seller aims to maximize her expected cumulative revenues over a finite selling horizon. At each demand epoch, the arriving customer observes the set of substitute products with positive inventory together with their prices. Based on this information as well as the customer’s own budget constraint, he either buys one of the available products, or leaves the system without making any purchase. We propose a choice model where a fixed ranking of the products is decided by the quality-price combination.

We show the monotonicity property of the optimal prices with respect to quality, inventory and time-to-go. We derive the distribution-free pricing methodology and obtain the robust bounds on the price increment through the first-order Taylor approximation. Our work also sheds light on the assortment design in terms of choosing the breadth of the product quality range as well as the number of products in the assortment.

Keywords: Dynamic pricing, demand substitution, consumer choice model, budget constraint, approximations.

1 Introduction

1.1 Motivation

In this paper, we study the firm’s dynamic pricing problems of differentiated but substitutable products. The firm aims to maximize her expected cumulative revenue over a finite selling season. The products could be differentiated in one or more attributes, but we aggregate these attributes into the “quality” index for the purpose of our analysis. The products are substitutable in the sense that customer can pick any product within the in-stock offering if the product has an appealing quality-price combination. Dynamic pricing is a valuable tool for products with short selling season and limited capacity. However, it is largely complicated by the noticeable substitution behavior by customers based on quality-price trade-offs. For example, in retailing sector, a clothing company many operate multiple brands that differ in design, material quality and fashion. How to make the promotional decisions for one brand during the seasonal or holiday sales depends on the brand’s own inventory level, how closed it is to the end of the selling season, as well as the substitution

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effect driven by the prices of all other brands. In the hotel example, there might be various types of rooms (e.g. standard v.s. deluxe rooms) that differs in the facilities and service available for the guests. In this case, the demand for an individual room type actually does not only depend on the price and non-price characteristics of that room type, but also on those of all room types. As a result, a firm must understand the choices that consumers make when facing such a product assortment and determine the prices for different products jointly.

Motivated by these revenue management applications, we aim to investigate how the product’s quality and inventory level affect the firm’s pricing decision in the stochastic environments. An important part of the study is to understand the consumer’s purchase behavior, that is, how customers pick a particular product within the family of substitute products.

We consider in this work two main sources of substitution. The first one, called inventory-driven substitution, is due to stockouts. If a product runs out of stock then part of the demand for that product will shift to a substitute product. The second source of substitution is referred to as the price-driven substitution. We assume that all products’ attributes, except for the price, are kept fixed over time. The pricing policy affects customer’s purchase decision in the following two ways. First, we assume that each consumer has a fixed budget, which is a distinctive feature of our choice model. With a limited budget, the customer may not be able to choose his favorite product from the full assortment; instead he will only consider those products within his budget constraint. In this way, the customer’s budget together with the price jointly determine his choice set. Second, the firm’s pricing policy would affect the final value or customer’s utility of the different substitute products. The consumer utility on the product is jointly determined by its quality and price.

In our model, quality is an exogenous factor and we assume that consumers have the same “taste”, that is, all customers would have the same preferences over the set of substitute products. For example, if a hotel sets the same price to all the room types, then consumers would prefer deluxe rooms over standard rooms. Different pricing policy may change the ranking of the products; however, the ranking is the same among customers and deterministic under any given pricing policy. This is another key feature that distinguishes our model from the random utility model that has been widely studied in the literature. The random utility models such as MNL or nested logit model assume a random ranking among products in the sense that a larger utility for some product only increases its probability of being ranked higher, given everything else unchanged. In contrast, we consider a setting where there is a perfect segmentation in the population due to customer’s self-selection.

Our model on vertically differentiated products that create a segmented market has applications in several industries that concerns with perishable products as well as the limited capacity. For example, it applies to hospitality (e.g. hotel rooms with different amenities and service), entertainment and sport (e.g. event tickets for different set locations), and information technology (e.g. 

2
advertisement slots at different positions of web pages) industries.

As a prelude of the results to come, we would like to highlight the following contributions and findings of our research. First, we develop a consumer choice model for substitutable products, where the quality and price jointly decide a deterministic ranking of products. The model captures in a parsimonious way the interplay between price and quality, and we are able to show that optimal prices increase in the quality levels. Another key feature is that we take into account the customer’s budget constraints, which is an important practical concern, however ignored by most of the models on consumer choice in the OM literature.

Second, we incorporate demand substitution effects and their impact on optimal pricing policies in a revenue management setting with multiple products. We allow for more than one spill-over events among products. Indeed, we show that with limited inventory, the price of one product decreases in the inventory levels of all the products. Therefore, the change of inventory level of one product will affect the purchase probability of any other products on offer.

Third, we propose a heuristic method to generate some simple and robust pricing rules. We derive the distribution-free bounds of the price increment between different products for both cases with unlimited and limited supply. Interestingly, with no inventory constraints, the bounds are also independent of the relative value of the quality level, and they only depend on the position of the products in the ranking.

Finally, the numerical study sheds light on how the products should be differentiated in the assortment. This includes determining the product quality range as well as the number of different products in the assortment. We find that products with low quality level should be excluded from the assortment. Our experiments also show that the expected revenue from ten products in the assortment generates 93% of the revenue if the firm could sell infinite number of different products. We believe these are important managerial insights for category managers in the retail industry.

1.2 Literature Review

In terms of the existing literature, there are two main streams of research that are closely related to our work: (i) consumer choice models and (ii) dynamic pricing models for multiple products with correlated demand. In what follows, we attempt to position our paper with respect to similar research without reviewing the vast literature in these areas.

In the recent decade, there has been a growing interest in the operations management (OM) literature in studying the consumer choice model, which has been extensively used in econometrics and marketing literature (e.g. [Train (1986) and Berry et al. (1995)]). This provides a specific way to model the individual purchase behavior and new perspective to model demand correlation. Fundamentally, the customer’s choice depends critically on the set of available products and can be modeled using a discrete choice framework. This may be a general choice model or may also
be specialized to more commonly used models such as the multinomial logit (MNL) model. The early focus of research based on consumer choice model in OM literature is on the assortment decision or the inventory control decision. van Ryzin and Mahajan (1999) are the first to adopt the MNL model to determine the demand distribution for substitute products and to study the optimal assortment. Talluri and van Ryzin (2004), and Zhang and Cooper (2005) consider the firm’s dynamic capacity control policies for airline revenue management. Focusing on a single-leg yield management problem with exogenous fares, Talluri and van Ryzin (2004) study how consumers purchase behavior affects the booking limits for various fare classes. Zhang and Cooper (2005) extend their model and consider capacity control for parallel flights. Joint inventory and static pricing policy for a given assortment has been studied in Aydin and Porteus (2008), where the authors define a broader class of demand models that includes the MNL model as its special case.

Consumer choice models that lead to market segmentation have been studied in the marketing and economics literature since decades ago. Early examples include Mussa and Rosen (1978) and Moorthy (1984), where they study the firm’s static optimal pricing policy for vertically differentiated products, and customer’s purchase decision is based on both price and quality of the product. In the OM literature, market segmentation has been considered in the context of remanufacturable products. Customers choose between the new and re-manufactured products which have different prices and qualities. The firm aims to maximize over her pricing policy and other operation decisions such as whether to remanufacture the product and choosing the technology level for remanufacturing (see for example Ferrer (2000) and Debo et al. (2005)). To our knowledge, we have not seen any paper that considers consumer choice model that leads to market segmentation in a revenue management context.

The second stream of literature related to our work corresponds to the study of intertemporal pricing strategies with stochastic demand. Starting from the seminal paper by Gallego and van Ryzin (1994), the revenue management community has focused its attention on the problem of how to dynamically adjust the pricing policy for a limited capacity of products over a finite selling horizon. The early literature is vast and we refer readers to the comprehensive surveys by Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003) and Talluri and van Ryzin (2004).

In the study on dynamic pricing models for multiple products, pricing decisions are jointly decided for all alternatives because of joint capacity constraints and due to demand correlations. The problem of dynamic pricing for multiple products was first investigated by Gallego and van Ryzin (1997) in the context of network revenue management. Due to the difficulty in solving the multi-dimensional optimization problem and systems of differential equations, the authors propose two heuristic policies by solving the deterministic counterpart of the problem, which are shown to be asymptotically optimal. In Maglaras and Meissner (2006), the authors study a similar model in a single-resource-multiproduct setting, a special case of Gallego and van Ryzin (1997). They
show that the optimal pricing policy for one product is affected by the other products through
the aggregate demand, and therefore they are able to reduce the multi-dimensional problem to a
one-dimensional problem. These papers are examples of dynamic pricing with a general model on
demand correlation, without explicitly modeling the individual consumer choice.

The monopolist’s dynamic pricing problem under consumer choice models has attracted sig-
nificant attention in the recent OM literature. However, due to the complexity of analysis and
for mathematical tractability, these papers all use the MNL model to characterize consumer choice
(e.g. Dong et al. (2009), Akçay et al. (2010), Suh and Aydin (2011), Li and Graves (2012)). Among
these works, Dong et al. (2009) and Akçay et al. (2010) study the pricing problem for perishable
products which are differentiated by quality levels. Through numerical experiments, Dong et al.
(2009) demonstrate that dynamic pricing offers great value when inventory is scarce or when the
quality range of the products is wide. Akçay et al. (2010) study both cases where products are
vertically and horizontally differentiated. They show similar results on vertically differentiated
products to ours, that is, the optimal prices are monotone in time-to-go, inventory and quality
levels, which may not hold for horizontally differentiated products.

Unlike our paper, none the above work considers the market where there is a perfect segmenta-
tion of customer population. Also, as we have mentioned, another distinctive feature of our choice
model is that we assume that consumers have a fixed budget which limits their purchasing deci-
sions. In general, many papers using consumer choice models in the OM literature do not model
the customer’s budget constraint, in that it is always assumed high price can be compensated by
the high quality of the product, and it is the final utility (e.g. the difference between the quality
level and price) that determines customer’s purchase behavior. A notable exception is the work by
Hauser and Urban (1986) who study a budget constraint consumer model closely related to ours.
However, there are some important differences between Hauser and Urban’s model and ours. We
postpone this discussion to section Section 2 where we spell out the details of our choice model.

2 Demand Model

2.1 Arrival process

In this section we present the specific choice model that we use to characterize customers’ purchasing
behavior. Let \( S \triangleq \{1, 2, \ldots, N\} \) be a family of substitute products. We define for this family the
cumulative demand process \( D(t) \). For the purpose of our pricing model, we assume that this
cumulative demand is independent of the price vector chosen by the retailer. In other words, at
each moment in time there is a fixed demand intensity of potential buyers that are willing to
purchase a product within the family \( S \).

In order to fit the data to this type of model, we need to understand the nature of \( D(t) \). A
variety of different approaches can be used to model the cumulative demand process \( D(t) \). For instance, the total demand can be modeled as a deterministic process using seasonality data. We can also try to fit a stochastic process such as a non-homogeneous Poisson process. A more static approach would be to consider that the demand \( D(t) \) for the next \( t \) days (e.g. a week) is normally distributed with mean \( \mu(t) \) and variance \( \sigma(t) \). For the purpose of this paper however, we will model cumulative demand \( D(t) \) as a time-homogeneous Poisson process with intensity \( \lambda \).

On the other hand, the specific choice that an arriving customer makes does depend on prices. In particular, we assume that given a vector of prices \( p_S(t) = \{ p_i(t) : i \in S \} \) a particular buyer purchases product \( i \in S \) with probability \( q_i(p_S(t)) \). We denote by \( q_0(p_S(t)) \) the non-purchase probability. It is assumed that upon arrival, a customer either buys certain product from \( S \), or he leaves the market with no purchase. Observe that the probability function \( q_i(\cdot) \) is time-invariant and only depends on the price vector. Hence, the incremental demand for product \( i \) at time \( t \) satisfies \( dD_i(p_S(t)) = q_i(p_S(t))dD(t) \) for all \( i \in S \). We will discuss in more details on how the consumer choice probability \( q_i(p_S(t)) \) is derived.

### 2.2 Consumer choice model

As we discussed in Section 1, the literature on customer’s choice model is extensive and has looked at the problem of modeling the \( q_i(p_S(t)) \)’s from various different angles. One of the most commonly used models is the MNL model (introduced by Luce (1959)). The MNL assumes that every consumer will assign a certain level of utility to each product, and will select the one with the highest utility level. To capture the lack of knowledge that the seller has about the population of potential clients, and their inherent heterogeneity, the MNL models the utility of each product as the sum of a nominal (expected) utility, plus a zero-mean random component representing the difference between an individual’s actual utility and the nominal utility. When these stochastic components are modeled as i.i.d random variables with a Gumbel (or double exponential) distribution, the probability of selecting each product \( i \) is given by

\[
q_i(p_S) = \frac{\exp(u_i(p_S))}{\sum_{j \in S \cup \{0\}} \exp(u_j(p_S))},
\]

where \( u_i(p_S) \) is the utility of product \( i \) given the vector of price \( p_S \). The simplicity of the MNL model makes it appealing from an analytical perspective, however, it has some restrictive properties. In particular, it does not establish a single (absolute) ranking of the products based on non-price attributes such as quality or brand prestige. Thus, it is hard to incorporate customer segmentation using the MNL framework.

†In general, quantities with subscript \( S \) will be used to denote the corresponding vector; for instance the price vector at time \( t \) is \( p_S(t) = (p_1(t), \ldots, p_N(t)) \).
Hauser and Urban (1986) proposed a quite different consumer choice model that overcomes some of these limitations. Their model assumes each customer solves the following knapsack problem

\[
\max_{g_i, y, i \in S, y} \quad u_y(y) + \sum_{i \in S} u_i g_i \\
\text{subject to} \quad \sum_{i \in S} p_i g_i + y \leq w \quad \text{(MP2)} \\
y \geq 0 \quad g_i \in \{0, 1\} \quad \text{for all } i \in S,
\]

where \(w\) is the buyer’s available monetary budget, \(p_i\) is the price of product \(i\), \(u_i\) is the utility of product \(i\), and \(u_y(y)\) is the utility associated to \(y\) units of cash. In words, MP2 models the choice problem of a buyer who wants to select a subset of products from the set \(S\) to maximize his utility by taking into account his limited budget \(w\). In addition, the buyer’s utility for a fixed bundle of products is the sum of the utilities of the products in the bundle plus a residual utility for any unused cash.

In this paper, we assume that the customer type is characterized by a pair \((w, u_0)\), where \(u_0\) represents his reservation utility or non-purchase utility and \(w\) represents the customer’s budget. We also suppose that there is a common ranking across all buyers of the products in \(S\) based on their intrinsic utilities \(u_i\). It is assumed that these arriving buyers are homogeneous on their valuation of the product \(u_i\) for all \(i\) but heterogeneous on their reservation utility \(u_0\) and budget \(w\).

For a given type \((w, u_0)\), the consumer choice problem is modeled using a specialized version of Hauser and Urban’s MP2 model with the following characteristics.

1) First, we assume that every customer is willing to buy at most one unit from the set \(S\). This assumption specializes MP2 to the case of substitute products.

2) We also extend the set of physical products \(S\) with a non-purchase option, that we denote by product 0, with intrinsic utility \(u_0\). We will refer to \(u_0\) as the non-purchase or reservation utility.

3) Finally, we assume that a consumer that buy product \(i \in S\) get a net utility \(U_i \triangleq u_i + \alpha (w - p_i)\), for a constant \(\alpha > 0\) which captures the marginal value of the residual budget after a purchase (if any) is made. For completeness, we let \(U_0 \triangleq u_0 + \alpha w\) be the net utility of the non-purchasing option.

Under the three conditions above, a customer with type \((w, u_0)\) will solve the following utility
maximization problem to decide his purchasing behavior, given the options in $S$ and prices $p_S$.

\[
\max_{x_0,x} \quad (u_0 + \alpha w) x_0 + \sum_{i \in S} (u_i + \alpha (w - p_i)) x_i \\
\text{subject to} \quad \sum_{i \in S} p_i x_i \leq w \\
\qquad \quad x_0 + \sum_{i \in S} x_i = 1 \\
\qquad \quad x_i \in \{0, 1\} \quad \text{for all } i \in S \cup 0,
\]

(MP2-1)

It is worth noticing that condition 3) has specialized the residual utility function $u_y(y)$ in MP2 to be quasi-linear in wealth, that is, $u_y(y) = \alpha y$. An important implication of this quasi-linear utility is that it defines preferences that are independent of the consumer’s budget. Indeed $U_i \geq U_j$ if and only if $u_i - \alpha p_i \geq u_j - \alpha p_j$. Therefore, once the price vector is given at each time point, the preference of the products is fixed and the same for customers with different budget. However, the budget does determine the subset of products that the consumer can afford to buy.

We will refer to our model as MP2-1 where the ‘1’ is used to emphasize that a customer buys at most one unit. It is interesting to note that MP2 corresponds to a Knapsack problem, for which there is no efficient (polynomial) algorithm known, while the MP2-1 model can be easily solved with a greedy algorithm (more on this below). In terms of applications, we expect the MP2-1 model to be adequate for modeling the demand for durable products such as refrigerators, personal computers, household-electric equipment, among many others, where customers usually assign a certain budget for the product, and buy only a single unit. In this respect, our model is not intended to capture the problem of multi-category choice behavior faced by retailers such as groceries and supermarket purchases.

To fully characterize the customer’s purchase decision, we index the products in $S$ in descending order of their utilities, that is, $u_1 > u_2 > \cdots > u_N$, where $N \triangleq |S|$ is the total number of products in $S$. Note that assuming that every customer has the same ranking over the $\{u_i\}$ implies that – in the absence of price considerations – every customer prefers product $i$ to product $j$ if $i < j$. For example, we can think of $u_i$ as proxy for the quality offered by product $i$. We expect this assumption to hold for customers belonging to the same market segment. We also assume that the ordering induced by the $u_i$’s is common knowledge to the consumers and the seller.

Then we are able to solve the MP2-1 problem for customer with type $(w, u_0)$ using a greedy algorithm. We search the list of products sequentially starting from product 1. We stop as soon as we find a product $i$ with price $p_i \leq w$ and $u_i - \alpha p_i \geq u_0$, in which case the customer will purchase this product. If $w < p_i$ and $u_0 > u_i - \alpha p_i$ for all $i$, then the customer will not make any purchase. Note that this algorithm assumes that the following two conditions are satisfied: (i) $p_1 > p_2 > \cdots > p_N$, and (ii) $u_i - \alpha p_i > u_j - \alpha p_j$ for all $i < j$. We denote by $P_S$ the set of
prices that satisfy conditions (i) and (ii). Condition (i) formalizes the intuition that products with higher intrinsic utilities should sell at a higher price. On the other hand, condition (ii) guarantees that products with higher intrinsic utilities \( u_i \) generate higher net utilities \( U_i = u_i - \alpha p_i \). In Section 3, we will formalize these two conditions. Figure 1 shows schematically the segmentation of the customers’ population among the different products on the \((w, u_0)\) space under the above two conditions.

![Figure 1: Customers’ segmentation in the \((w, u_0)\) space. For example, product 2 is purchased by all those consumers that have \( u_0 \leq u_2 - \alpha p_2 \) and \( p_2 \leq w < p_1 \).]

2.3 The aggregate demand process

From the seller’s perspective, the demand process is the combination of the external arrival process of consumers \( D(t) \) and the MP2-1 problem solved by each of these consumers at their arrival epoch given the vectors of prices and available inventories.

As mentioned, we assume that the consumers are heterogeneous in their budget and reservation utility \((w, u_0)\). Specifically, we assume that \((w, u_0)\) is distributed among the population of buyers according to a probability distribution \( F(p, u) \triangleq P(w \leq p \text{ and } u_0 \leq u) \). This joint probability distribution allows us to model the positive correlation between \( w \) and \( u_0 \) that we expect to observe in practice \( (i.e., \) customers with higher reservation utility tend to have a higher budget).

Based on the MP2-1 choice model, we can compute the probability \( q_i(p_S) \) that an arriving consumer chooses product \( i \) given the vector of prices \( p_S \). For a given distribution \( F \), and a given
price vector $p_S \in \mathcal{P}_S$, we have that

$$q_i(p_S) = q_i(p_{i-1}, p_i) = F(p_{i-1}, u_i - \alpha p_i) - F(p_i, u_i - \alpha p_i) \quad \text{and} \quad q_0(p_S) = 1 - \sum_{i \in S} q_i(p_{i-1}, p_i), \quad (1)$$

where we set $p_0 \triangleq \infty$. From a pricing perspective, we note that the probability of purchasing product $i$ depends exclusively on $p_i$ and the price of the next alternative $p_{i-1}$. Interestingly, this model is not sensitive to equivalent alternatives, and by construction, fully incorporates the notions of product differentiation and demand segmentation. Observe that when $\alpha = 0$, the previous expression simplifies to

$$q_i(p_{i-1}, p_i) = F(p_{i-1}, u_i) - F(p_i, u_i). \quad (2)$$

Furthermore, when $N = 1$ (i.e., there is only one product), then according to (2) the probability that a customer buys the product at price $p$ is equal to

$$q(p) = F(\infty, u) - F(p, u).$$

Notice that in this $\alpha = 0$ and $N = 1$ situation when $u = \infty$ (i.e., the perceived utility associated with the product is very high), the fraction of customers that buy the product is given by $q(p) = 1 - F(p)$. The distribution function $F$, in this single-product case, characterizes the distribution of the reservation price\(^\dagger\) that the population of consumers has for that particular product. Thus, we view (2) as a generalization of the notion of reservation price to a multi-product setting.

For the sake of mathematical tractability, throughout this paper we will assume that $F$ satisfies the following assumption.

**Assumption 1** The probability distribution $F(p, u)$ is strictly increasing\(^\ddagger\) and twice continuously differentiable. For every $u$, $p_1$ and $p_2$ with $u - \alpha p_2 \geq 0$, $F_p(p_1, u - \alpha p_2) - \alpha p_2 F_p(p_1, u - \alpha p_2) \geq 0$.

For fixed $u$, the function $F(p, u) + p F_p(p, u)$ is unimodal in $p$ and converges to $F(\infty, u)$ as $p \uparrow \infty$.

The notation $F_p(p, u)$ stands for the partial derivative of $F(p, u)$ with respect to $p$. The assumption about the smoothness and monotonicity of $F$ are rather standard and they are satisfied by most common bivariate distribution functions such as bivariate normal or bivariate Weibull. The assumptions on the auxiliary function, $F(p, u) + p F_p(p, u)$, is required to guarantee the existence of an optimal pricing policy in section 3. Again, this condition is not particularly restrictive.

\(^\dagger\)The reservation price is the maximum price that a customer is willing to pay for a product in a single-product setting. See Bitran and Mondschein (1997) for details about the use of reservation price distributions in pricing models.

\(^\ddagger\)Throughout the paper, we shall use the terms, increasing and decreasing, in the nonstrict sense, to represent nondecreasing and nonincreasing, respectively. Otherwise, we use the words ‘strictly’ increasing or decreasing.
Having derived the consumer choice probability \( q_i(p_S) \), from the seller perspective, the incremental aggregate demand for product \( i \) at time \( t \) then satisfies \( dD_i(p_S(t)) = q_i(p_S(t))dD(t) \) for all \( i \in S \).

The rest of the paper is organized as follows. In section 3, we first study the case where there is infinite inventory for all products. In section 4, we relax this condition. Specifically, we formulate the problem where the consumer’s substitution behavior is both price-driven and stock-out driven. We will show how the proposed choice model extends to the limited supply case. We then study the fluid model of this general case in section 5.

3 Unlimited Supply Case

In this section, we analyze optimal pricing strategy under the assumption that inventory are sufficiently large so there is never stock-outs, in which case, the consumer substitution behavior is purely price-driven. In what follows, we first propose an efficient line-search algorithm to compute the optimal pricing policy and show the sufficient and necessary conditions for the existence of the optimal solutions. We then apply the first-order approximation on the optimality condition for the pricing policy to generate a pricing heuristic as well as the bounds of the price ratios of different products. This serves as a good guidance in practice when managers need to decide the extent of markup or markdown between the two products with different quality levels. Finally, we study the case where there are infinitely many different products in the assortment and the quality level is continuously changing. Although this limiting case is rarely seen in practice, we do obtain important insights on how differentiated the products should be in one assortment, which is a crucial question for the category managers. The decision includes the range of product quality as well as the number of products in the assortment.

3.1 Optimal Pricing Policy

In this unlimited supply case, to solve the seller’s optimal pricing problem, we simply maximize the expected revenue rate in each time instant. Then the optimal price \( p_S \) is constant over time. Let \( D \) be the cumulative number of customers arriving during the entire horizon. Thus, conditioned on the value of \( D \), the total expected revenue associated to a price vector \( p_S \) can be written as \( DW(p_S) \), where \( W(p_S) \) is the expected revenue rate given by

\[
W(p_S) = \sum_{i=1}^{N} p_i q_i(p_S)
\]

and the resulting optimization problem in this unlimited supply case reduces to

\[
\max_{p_S \in \mathbb{R}^N} W(p_S). \tag{3}
\]
We first show that to search for the optimal price $p^*_S$, we could restrict ourselves to the set $P_S$, which is formalized in the following proposition.

**Proposition 1** Consider the problem of pricing $N$ products with unlimited supply. The optimal price vector $p^*_S$ belongs to $P_S \triangleq \{ p_S : p_i+1 < p_i < p_i+1 + \psi_{i+1}, \text{ for all } i = 1, \ldots, N-1 \}$, where $\psi_i \triangleq \frac{u_{i-1} - u_i}{\alpha}$ for all $i = 2, 3, \ldots, N$.

From Proposition 1 and by the nature of MP2-1 choice model, we could rewrite the seller’s problem in (3) as

$$\max_{p_S \in P_S} W(p_S) = \sum_{i=1}^{N} p_i q_i(p_i, u_i).$$

(4)

### 3.1.1 Algorithm for Computing Optimal Pricing Policy

In this subsection, we will limit our analysis to the case where the value of residual money is equal to zero, i.e. $\alpha = 0$. In the non-zero $\alpha$ situation, however, $q_i(p_S)$ involves the integration over a non-rectangular area, which under a general probability distribution, $F(p, u)$, has no closed form expression. We believe this simplifying assumption, which certainly benefits mathematical tractability, adequately represents situations where customers make decisions in a local way, i.e. not considering all the alternative uses they could give to their money. For example, it is likely that someone who needs to buy a refrigerator will assign a certain budget for the acquisition of this item, and will probably not be analyzing all the possible uses he/she could give to that budget.

In what follows, we provide an algorithm to compute the optimal pricing policy to problem (4), by using a line-search procedure. We will also show the sufficient and necessary conditions for the existence of the optimal solution.

The first-order optimality conditions of problem (4) are given by

$$F(p_i-1, u_i) = F(p_i, u_i) + p_i F_p(p_i, u_i) - p_i+1 F_p(p_i+1, u_i) \quad \text{for all } i = 1, \ldots, N.$$  

(5)

Equation (5) has an interesting interpretation. To see this, let us first multiply both sides by $dp_i$, and then rearrange the terms as follows,

$$dp_i q_i(p_i, u_i) = p_i dp_i F_p(p_i, u_i) - p_i+1 dp_i F_p(p_i+1, u_i+1).$$

The left-hand side corresponds to the incremental expected revenue obtained by increasing the price of product $i$ by $dp_i$. The right-hand side is the associated expected cost of this price increment. The first term of the right-hand side is the lost revenue due to the fraction of customers who were willing to buy $i$ at the initial price, but are not willing to buy it at the higher price. However, since the retailer offers a less expensive product $i + 1$, some of these customers will switch and buy this
less expensive product $i+1$, allowing the seller to recover part of the lost benefits. This effect is captured by the second term of the right-hand-side.

In general, one needs to solve equations in (5) which is a multidimensional system of nonlinear equations. Fortunately, it turns out that a simple one-dimensional search can be set to solve it efficiently because of its diagonal structure. We solve the system equations backwards. With $i = N$, equation (5) becomes

$$F(p_{N-1}, u_N) = F(p_N, u_N) + p_N F_p(p_N, u_N).$$

Therefore, fixing $p_N = \bar{p}$ we can solve for $p_{N-1}$ as a function of $\bar{p}$. The value of $p_{N-1}$, as a function of $\bar{p}$, is uniquely determined by

$$p_{N-1}(\bar{p}) = F^{-1}\left(F(\bar{p}, u_N) + p \cdot F_p(\bar{p}, u_N), u_N\right). \quad (6)$$

Note that $F^{-1}(\cdot, u)$ is the inverse function of $F(p,u)$ with respect to $p$ for a fixed $u$. By Assumption 1, this inverse function $F^{-1}(x,u)$ is well defined for $x \in [0, F(\infty, u))$. Therefore, our choice of $\bar{p}$ must by restricted so that $F(\bar{p}, u_N) + p \cdot F_p(\bar{p}, u_N) < F(\infty, u_N)$. Let us define

$$\bar{p}^\text{max}_N \triangleq \sup\{p \geq 0 : F(p, u_N) + p \cdot F_p(p, u_N) < F(\infty, u_N)\}.$$ 

Assumption 1 guarantees that the condition $F(p, u_N) + p \cdot F_p(p, u_N) < F(\infty, u_N)$ is satisfied if and only if $p < \bar{p}^\text{max}_N$. Therefore, we can restrict the choice of $\bar{p}$ to the interval $[0, \bar{p}^\text{max}_N]$. Note that $p_{N-1}(\bar{p})$ is increasing in $\bar{p}$ with $p_{N-1}(0) = 0$ and $p_{N-1}(\bar{p}^\text{max}_N) = \infty$.

Similarly, we can sequentially (backward on the index $i$) solve equation (5) for all $p_i$ as a function of $\bar{p}$, that is,

$$p_i(\bar{p}) = F^{-1}\left(F(p_i(\bar{p}), u_i) + p_i(\bar{p}) \cdot F_p(p_i(\bar{p}), u_i) - p_{i+1}(\bar{p}) \cdot F_p(p_i(\bar{p}), u_{i+1}), u_i\right), \forall i = N - 1, \ldots, 2.$$ 

For each $i$, we need to guarantee that the argument of $F^{-1}$ is bounded from above by $F(\infty, u_i)$. In other words, we have to restrict the choice of $\bar{p}$ such that

$$F(p_i(\bar{p}), u_i) + p_i(\bar{p}) \cdot F_p(p_i(\bar{p}), u_i) - p_{i+1}(\bar{p}) \cdot F_p(p_i(\bar{p}), u_{i+1}) < F(\infty, u_i), \quad \text{for all } i = N - 1, \ldots, 2. \quad (7)$$

For an arbitrary distribution $F$, the left-hand side can be a complicated function of $\bar{p}$ and in general, we have not been able to prove this unimodal property under Assumption 1. Thus imposing this inequality condition is not straightforward. However, all the computational experiments that we have performed using bivariate distributions such as normal, Weibull, and exponential have shown this property. In what follows, we assume that left-hand side of (7) is a unimodal function of $\bar{p}$. Then, as before, we can show that $\bar{p}$ must be restricted to a closed interval of the form $[0, \bar{p}^\text{max}_i]$. Furthermore, the solution $p_i(\bar{p})$ is increasing in $\bar{p}$ with $p_{i-1}(0) = 0$ and $p_{i-1}(\bar{p}^\text{max}_i) = \infty$. 

Finally, the condition for \( i = 1 \) is used for checking optimality. That is, if

\[
F(p_0, u_1) = F(p_1(\bar{p}), u_1) + p_1(\bar{p}) F_p(p_1(\bar{p}), u_1) - p_2(\bar{p}) F_p(p_1(\bar{p}), u_2)
\]

holds then the solution \( p_S(\bar{p}) \triangleq (p_1(\bar{p}), p_2(\bar{p}), \ldots, p_N(\bar{p})) \) satisfies the optimality condition in \( \text{[5]} \), if not we change the starting point \( \bar{p} \) and iterate. The following algorithm formalizes this procedure.

**Unlimited Inventory Algorithm:**

**Step 1:** Set \( p_{N+1} = 0, p_N = \bar{p}, \) and \( p_0 = \infty. \)

**Step 2:** Solve recursively the system

\[
F(p_{i-1}, u_i) = F(p_i, u_i) + p_i F_p(p_i, u_i) - p_{i+1} F_p(p_i, u_{i+1})
\]

to compute \( p_i, i = N - 1, \ldots, 1. \)

**Step 3:** Compute

\[
\eta = F(p_0, u_1) - F(p_1, u_1) - p_1 F_p(p_1, u_1) + p_2 F_p(p_1, u_2).
\]

If \(|\eta| \leq \epsilon, \) for some pre-specified \( \epsilon, \) then stop; the solution \( (p_1, \ldots, p_N) \) is an \( \epsilon \)-solution. If \( \eta > \epsilon \) then \( p_N \leftarrow p_N + \delta, \) otherwise \( p_N \leftarrow p_N - \delta. \) Go to Step 2 and iterate. \( \Box \)

It is straightforward to show that for every \( \bar{p} \) the solution \( p_S(\bar{p}) \) belongs to \( P_S \), which is consistent with the optimality condition identified in Proposition 6.

To ensure that the previous algorithm is well defined, we need to address the problem of existence of a solution to the first-order optimality conditions in \( \text{[5]} \). The following result identifies sufficient and necessary conditions for the existence of a solution as well as a set of bounds for this solution. Let us define two auxiliary functions

\[
L(p, u, \hat{u}) \triangleq F(p, u) + p \left( F_p(p, u) - F_p(p, \hat{u}) \right) \quad \text{and} \quad U(p, u) \triangleq F(p, u) + p F_p(p, u).
\]

**Proposition 2 (Sufficient and Necessary conditions)**

A sufficient condition for the existence of a solution to \( \text{[5]} \) is that there exists a price \( \hat{p} \) that solves

\[
L(\hat{p}, u_1, u_2) = F(p_0, u_1),
\]

while a necessary condition for the existence of a solution is that there exists a \( \tilde{p} \) such that

\[
U(\tilde{p}, u_1) = F(p_0, u_1).
\]

In addition, every solution \( (p_1^*, \ldots, p_N^*) \) to \( \text{[5]} \) satisfies \( p_i^{\min} \leq p_i^* \leq p_i^{\max} \), where the sequence of

\[\text{\footnotesize \text{\footnotesize This conclusion follows using induction over } i = N, N - 1, \ldots, 1 \text{ and the monotonicity of } F(p, u) \text{ with respect to } p.}\]
lower and upper bounds is computed recursively as follows:

$$p^\text{min}_i = \text{argmin} \{ p : F(p^\text{min}_{i-1}, u_i) = U(p, u_i) \} \text{ and}$$

$$p^\text{max}_i = \text{argmax} \{ p : F(p^\text{max}_{i-1}, u_i) = L(p, u_i, u_{i+1}) \}$$

(9)

(10)

with boundary conditions $p^\text{max}_0 = p^\text{min}_0 = p_0 = \infty$.

The sufficient condition identified in Proposition 2 is not particularly restrictive and most of
the distributions commonly used in practice satisfy it. One important case in this group is the
bivariate normal distribution. Figure 2 plots the functions $L(p, u_1, u_2)$ and $U(p, u_1)$ and shows how
to identify the lower and upper bounds, $p^\text{min}_1$ and $p^\text{max}_1$, respectively, for the case in which $F(p, u)$ is
a bivariate normal distribution.

Figure 2: Shape of $L(p_1, u_1, u_2)$ and $U(p, u_1)$ with $u_1 = 15, u_2 = 12$ for the case when $(w, u_0)$ has a bivariate normal distribution with mean $(10,10)$, variance $(2.0,1.0)$, and coefficient of correlation $\rho = 0.8$.

3.1.2 Numerical study

We next present a set of numerical experiments to show the behavior of optimal prices and revenues
under different settings. To model customers’ type, we consider a bivariate normal distribution with
mean $(\mu_w, \mu_{u_0}) = (1,1)$ and variance $(\sigma^2_w, \sigma^2_{u_0}) = (0.5,0.4)$. The quality of products is assumed to
be evenly distributed over $(0.5,3)$.

Our first analysis studies the effect of correlation between customers’ budget $(w)$ and their
non-purchasing utility $(u_0)$ over pricing policies. Figure 3 (a) shows optimal pricing policies for a
family of 20 substitute products for a set of four different $\rho$’s $(\rho = 0, 0.5, 0.7, 0.9)$. As presented in
this plot, optimal prices raise with the magnitude of the coefficient of correlation.

When the coefficient of correlation grows, the seller increases her ability to segment customers
according to their budget. Under high correlation settings, the quality of products serves the
seller as a proxy for customers’ budget, in the same way as time is used as a proxy for customers’
disposition to pay in the airline industry. In the limit, when \( \rho \uparrow 1 \), the seller knows with certainty that low quality products will be demanded only by low budget customers, whereas high budget customers will exclusively demand high quality products. This segmentation ability allows the seller to discriminate customers according to their budget, making it possible to increase prices so as to charge each customer type as much as he can pay.

Figure 3 (b) presents the effect of the number of choices over optimal prices. When the number of different products increases, the prices for low quality choices get reduced while for the high quality choices the opposite occurs. As the number of choices \( N \) goes to infinity, all the products, except for those with quality near \( u_{\text{max}} \), will have prices equal to zero. To better understand the phenomenon from this asymptotic case, note that when \( N \) grows very large, the price is close to zero for products with quality \( u \leq u_{\text{max}} - \epsilon \) for a small value of \( \epsilon \), and price grows rapidly for products with quality in range \([u_{\text{max}} - \epsilon, u_{\text{max}}]\). This implies that when the firm is able to offer infinitely many products, she is capable to provide customers who have different budgets with the products of (almost) the same quality \( u_{\text{max}} \). Thus the customers will be perfectly screened only according to their budgets. The price range should be the range of the customers’ budget and all customer type \((w, u_0)\) will be served. As \( N \) goes to infinity, the demand for a product of quality \( u \) is \( DF_p(p(u), u) \, dp(u) \). By Proposition 1, the optimal price strictly increases in the product quality, so the function \( p(u) \) can be inverted and then total revenues will be given by \( \int_{0}^{\bar{w}} p \, DF_p(p, u(p)) \, dp \), where \( \bar{w} \) is the upper bound of customer’s valuation. Since \( u(p) = u_{\text{max}} \) for all \( p \) in this asymptotic
case, the firm’s total revenue is

\[
\int_0^{\bar{w}} p \, dF_p(p, u(p)) \, dp = \int_0^{\bar{w}} p \, dF_p(p, u_{\text{max}}) \, dp = D \int_0^{\bar{w}} p \, dF(p, u_{\text{max}}) = D \cdot E[w]
\]

where \( E[w] \) is the expected value of customer’s budget.

Variations in the coefficient of correlation and the number of different products available do not
only influence pricing policies, but also have an important impact over total revenues. Our next
two computational experiments study these issues. We consider a fixed selling horizon in which the
retailer faces an average arrival of 100 customers during the whole period \((D = 100)\).

![Figure 4: (a) Optimal revenues as a function of coefficient of correlation \( \rho \); (b) Optimal revenues as a function of
the number of substitute products.](image)

Figure 4 presents the influence of the correlation coefficient over total revenues. When corre-
lation increases, it is possible to increment prices without reducing the number of non-purchasing
customers. This explains the increasing effect of \( \rho \) over revenues presented in the plot. It is inter-
esting to note however, that the level of correlation has only a limited influence over total revenues.
In this particular example, the impact is less than 2%.

### 3.2 Pricing Heuristics using First-Order Taylor Approximation

To further analyze the structure of the optimal pricing policy and obtain more insights, in this
section we use a first-order Taylor approximation in (5). As we shall see, this approximation provides
a clean and recursive way to compute the prices. We acknowledge that the pricing policy from this
heuristic may not be optimal in general. However, this does not prevent us from using the heuristic
to generate a feasible pricing policy. In what follows, we first characterize the property of the
heuristic price and then we compare its performance with the optimal pricing policy. We will show
through numerical experiments that the heuristic provides an efficient and robust (distribution-free) methodology to establish prices in a setting of substitute products.

Recall from Assumption 1 that \( F^{-1}(p, u) \) denotes the inverse function of \( F(p, u) \) with respect to \( p \) for a fixed \( u \). Then, based on equation (5) we get that

\[
p_{i-1} = F^{-1}(F(p_i, u_i) + p_i F_p(p_i, u_i) - p_{i+1} F_p(p_i, u_{i+1}), u_i)
\approx F^{-1}(F(p_i, u_i), u_i) + F_p^{-1}(F(p_i, u_i), u_i) \left( p_i F_p(p_i, u_i) - p_{i+1} F_p(p_i, u_{i+1}) \right)
= 2 p_i - p_{i+1} \frac{F_p(p_i, u_{i+1})}{F_p(p_i, u_i)},
\]

where the approximation follows from the first-order expansion of \( F^{-1}(x, u_i) \) around \( x = F(p_i, u_i) \) with \( F_p^{-1}(p, u) \) denoting the partial derivative of \( F^{-1}(p, u) \) with respect to \( p \). The second equality follows from the identity \( F_p^{-1}(F(p, u), u) F_p(p, u) = 1 \). We note that the approximation is exact for the case in which the budget \( w \) (or reservation price) is uniformly distributed and independent of the reservation utility \( u_0 \) (see Example 1 below).

The following proposition formalizes the structure of the heuristic price given in (11).

**Proposition 3** Suppose a pricing policy \( p_S \) satisfies the recursion in (11), then

(i) the relative mark-up of the price of product \( i \) with respect to the price of product \( i+1 \), \( \beta_i \triangleq \frac{p_i}{p_{i+1}} \), is bounded by \( 1 + \frac{1}{N-1} \leq \beta_i \leq 2 \), for \( i = 1, \ldots, N-1 \);

(ii) the absolute price differential, \( p_i - p_{i+1} \), is decreasing in \( i \), for \( i = 1, \ldots, N-1 \).

In words, Proposition 3 states that it is never optimal to mark-up more than twice the price of a product with respect to the next “lower quality” product. On the other hand, the price differential between two consecutive products increases with the level of quality, that is, \( p_i - p_{i+1} \leq p_{i-1} - p_i \).

**Example 1: Independent Budget and Reservation Utility**

A particular case for which the assumptions of proposition 3 hold trivially occurs when the budget \( w \) and reservation utility \( u_0 \) are independent random variables and \( w \) is uniformly distributed. In this situation, there are two distribution functions \( G \) and \( H \) such that \( F(p, u) = G(p) H(u) \) and \( G(p_{i-1}) = G(p_i) + G_p(p_i) (p_{i-1} - p_i) \). Under this condition, the results in proposition 3 hold directly. Furthermore, in this situation (5) implies that \( p_i = A_i \bar{p} \) for \( i = 1, \ldots, N \), where the coefficients \( \{A_i\} \) satisfy the recursion

\[
A_{i-1} = 2 A_i - A_{i+1} \frac{H(u_{i+1})}{H(u_i)} \quad \text{for all } i = 1, \ldots, N,
\]

and boundary conditions \( A_{N+1} = 0 \) and \( A_N = 1 \).

\(^\dagger\)This linear approximation also hold if the optimal prices for the different products are relatively close to each other.
The sequence \( \{A_i : i = 0, \ldots, N\} \) is decreasing in \( i \). This follows directly from the fact that \( H(u_{i+1}) \leq H(u_i) \) since our ordering of the products satisfies \( u_{i+1} \leq u_i \). Moreover, using induction it is straightforward to show that

\[
2^{N-i} \left[ 1 - \frac{1}{4} \sum_{k=i+1}^{N-1} \frac{H(u_{k+1})}{H(u_k)} \right] \leq A_i \leq 2^{N-i}.
\]

Finally, \( \bar{p} \) solves the fixed-point condition

\[
\bar{p} = \frac{1 - G(A_1 \bar{p})}{(A_0 - A_1)G_p(A_1 \bar{p})}
\]  \hspace{1cm} (12)

In the special case where \( G(p) \) is uniformly distributed in \([p_{\text{min}}, p_{\text{max}}]\) then (12) implies that the optimal price strategy is given by

\[
p_i = \frac{A_i}{A_0} p_{\text{max}} \quad \text{for all } i = 1, \ldots, N.
\]

As we have already mentioned our MP2-1 model can be viewed as a generalization of the simple reservation price formulation for single product. Similarly, condition (12) generalizes condition (2) in Bitran and Mondschein (1997).

The simplicity of Proposition 3 is very attractive for a managerial implementation. For instance, the bounds on the relative mark-ups are distribution-free which make them particularly appealing in those cases where there is little or non information about the demand distribution.

In Table 1 we present a family of 10 substitute products under two different customer segmentation schemes: a bivariate Weibull distribution and a bivariate normal distribution. The first two columns of the table characterize the product according to its quality (utility). The following ten columns present the optimal price \( (p^*_i) \), the lower and upper bound for the optimal price \( (p^*_{i_{\text{min}}} \text{ and } p^*_{i_{\text{max}}}) \), the optimal price difference between two consecutive products \( (p^*_i - p^*_i+1) \), and the relative mark-ups \( (\beta^*_i) \) for each of the ten products under both segmentation settings.

These results show that under both distributions, optimal prices comply quite well with proposition 3 (i.e. price differentials, \( p^*_i - p^*_i+1 \), are decreasing in \( i \), and relative mark-ups move within the established bounds). However, in order to implement the results in proposition 3 some additional work is required. In particular, we need to be able to translate the suggested bounds on the relative mark-ups on actual price recommendations. For this, we first get an approximation on the relative mark-up for product \( i \) using a convex combination of the bounds computed in proposition 3.

\[
\Pr[X > x, Y > y] = \exp \left\{- \left[ \left( \frac{x}{\theta_x} \right)^{\gamma_x/\delta} + \left( \frac{y}{\theta_y} \right)^{\gamma_y/\delta} \right]^\delta \right\}
\]
Table 1: Numerical Optimization of 10 substitute products. Two distributions are considered: i) bivariate Weibull distribution with scale parameters $\theta_w = \theta_u = 1$, shape parameters $\gamma_w = \gamma_u = 1$, and correlation parameter $\delta = 0.5$; ii) bivariate normal distribution with mean $\mu_w = \mu_u = 1$, variance $\sigma^2_w = \sigma^2_u = 0.5$, and coefficient of correlation $\rho = 0.5$.

<table>
<thead>
<tr>
<th>Product</th>
<th>$u_i$</th>
<th>Bivariate Weibull</th>
<th>Bivariate Normal</th>
<th>$\beta_{i_{\text{min}}}$</th>
<th>$\beta_{i_{\text{max}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$p_i^* \quad p_i^{\text{min}} \quad p_i^{\text{max}} \quad p_i - p_{i+1}$</td>
<td>$p_i^* \quad p_i^{\text{min}} \quad p_i^{\text{max}} \quad p_i - p_{i+1}$</td>
<td>$\beta_i^*$</td>
<td>$\beta_i^*$</td>
</tr>
<tr>
<td>1</td>
<td>1.50</td>
<td>1.90 0.73 6.80 0.65 1.52</td>
<td>1.42 0.86 2.01 0.37 1.35</td>
<td>1.11 2.00</td>
<td></td>
</tr>
<tr>
<td>2</td>
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<td>1.25 0.32 5.21 0.36 1.40</td>
<td>1.05 0.45 1.68 0.25 1.31</td>
<td>1.13 2.00</td>
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</tr>
<tr>
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<td>0.89 0.15 4.11 0.23 1.36</td>
<td>0.80 0.24 1.43 0.19 1.31</td>
<td>1.14 2.00</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.17</td>
<td>0.65 0.07 3.27 0.17 1.34</td>
<td>0.61 0.13 1.22 0.15 1.32</td>
<td>1.17 2.00</td>
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<tr>
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<td>0.49 0.04 2.59 0.12 1.34</td>
<td>0.47 0.06 1.05 0.12 1.34</td>
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<tr>
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<td>0.06 0.00 0.32 - -</td>
<td>- -</td>
<td></td>
</tr>
</tbody>
</table>

is, for a fixed $a \in [0, 1]$, we define the approximated relative mark-up for product $i$ as

$$\tilde{\beta}_i(a) \triangleq a \left( 1 + \frac{1}{N-i} \right) + (1-a) \cdot 2.$$

From Table 1, we see that the lower bound on $\beta_i$ is more accurate than the upper bound. Hence, we expect $a$ to be closer to one. In our computation experiments below, we choose $a = 1$ and $a = 0.7$. We can think of more sophisticated rules to choose $a$ (e.g., making it a function of $i$) but we do not investigate this issue here.

The next step is to get an approximation for the price of product 1, which we denote by $\tilde{p}_1$. One possible approach, that we use in our computation experiments, is to consider the solution using a particular demand distribution such as the uniform (see Example 1). Alternatively, the seller might have some prior estimate of the value of $p_1$ based on past experiences or based on the prices set by competitors. Once $\tilde{p}_1$ has been determined, we can compute the prices of products 2, 3, ..., $N$ as follows

$$\tilde{p}_i = \frac{\tilde{p}_{i-1}}{\tilde{\beta}_{i-1}(a)} = \frac{\tilde{p}_1}{\tilde{\beta}_1(a) \tilde{\beta}_2(a) \cdots \tilde{\beta}_{i-1}(a)}; \quad i = 2, \ldots, N.$$

When selecting the value of $\tilde{p}_1$ (and therefore the price of all the products), the seller should consider other constraints which are not captured by our model, such as price bounds based on costs and competition.

In Table 2 we compare optimal revenues with those generated applying the pricing strategy $\tilde{p}_S$ derived above. To do so, we define $R$ as the ratio between the revenue obtained by using $\tilde{p}_S$ and the optimal revenue. In these numerical experiments, customers are characterized by a bivariate normal distribution with $\mu_w = \mu_u = 1$ and $\sigma_w = \sigma_u = 1$. Three different values of

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5In fact, we only need an approximation for the price of one of the $N$ products; for ease of exposition we consider product 1.
ρ are considered. We analyze a setting of 10 substitute products with their quality randomly distributed over $[\mu_{u0} - \sigma_{u0}, \mu_{u0} + \sigma_{u0}]$. For each of the cases studied, a set of 100 random instances of product quality were generated to compute the mean and standard deviation of $R$ ($R_{mean}$ and $R_{std}$, respectively).

We perform the analysis using two values of $a$ (1.0 and 0.7). The value of $\tilde{p}_1$ is obtained by using a bivariate uniform distribution approximation ($\tilde{p}_1 = p_1^{unif}$). This uniform distribution is given by

$$
\Pr[w \leq p, u_0 \leq u] = \left( \frac{p - (\mu_w - \sigma_w)}{2\sigma_w} \right) \left( \frac{u - (\mu_{u0} - \sigma_{u0})}{2\sigma_{u0}} \right).
$$

Note that $p_1^{unif}$ can be computed easily using the results in Example 1.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\tilde{p}_1 = p_1^{unif}$</th>
<th>$R_{mean}$</th>
<th>$R_{std}$</th>
<th>$a = 1.0$</th>
<th>$R_{mean}$</th>
<th>$R_{std}$</th>
<th>$a = 0.7$</th>
<th>$R_{mean}$</th>
<th>$R_{std}$</th>
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<tbody>
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<tr>
<td>0.5</td>
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<td>.0226</td>
<td>.9835</td>
<td>.0069</td>
<td>.9856</td>
<td>.0075</td>
<td>.9806</td>
<td>.0202</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>.8516</td>
<td>.0745</td>
<td>.9806</td>
<td>.0202</td>
<td>.9806</td>
<td>.0202</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Revenues applying the approximation $\tilde{p}_S$ versus optimal revenues.

Table 2 shows some interesting issues. In the first place, it is important to highlight the fact that all of the mean values of $R$ are above 0.85, and in most cases, $R_{mean}$ is above 0.90. It is also possible to observe that the value of $\alpha$ plays an important role in $R_{mean}$. Finally, note that the results when using $\tilde{p}_1 = p_1^{unif}$ are quite good, specially when $a = 0.7$. This last observation implies that prices can be set without knowing $F(p, u)$, only the first two moments $\mu_{u0}$, $\mu_w$, $\sigma_{u0}$, and $\sigma_w$ are required. Based on these results, we believe Proposition 3 provides an efficient and robust (distribution-free) methodology to establish prices in a setting of substitute products.

### 3.3 Continuous Assortment

In this section we study the asymptotic behavior of an optimal pricing policy as the number of products in $S$ grows large. This model is motivated by current trends in the retail industry that promote widening the variety of products offered by retailers. (See Kok and Fisher (2004)). In this section, we proceed with arbitrary $\alpha \in [0, 1]$, and do not restrict to $\alpha = 0$.

There are three important implications from the analysis of the continuous assortment. First, as we shall see, the firm does not need too many differentiated products in the assortment to generate a significantly higher revenue. Our experiments show that when the assortment includes only ten products, it will capture 93% of the revenue from the case where the number of differentiated products $N$ goes to infinity. The second insight is that it is not always optimal for the firm to expand the breadth in the assortment. Products with low qualities can be excluded from the assortment. The third implication from this analysis is that, for a fixed range of product quality,
when the number of products in the assortment goes to infinity, the probability of a positive demand for any product will converge to zero. This further implies that even with limited supply, as \( N \to \infty \), the inventory constraint drops out of the seller’s pricing problem. Therefore, as \( N \) grows large, the pricing problem with limited supply converges to the one with unlimited supply.

To formalize the asymptotic analysis that follows we need to introduce some additional notation. First, let us parameterize the firm’s problem by \( N \), the number of products in the assortment \( S(N) \). We find it convenient to index the product in \( S(N) \) by their intrinsic utility \( u \) so that product \( u \) is part of the retailer’s menu if \( u \in S(N) \). Let us define \( \underline{u} = \min\{u \in S(N)\} \), \( \bar{u} = \max\{u \in S(N)\} \), and \( \Delta(N) = (\bar{u} - \underline{u})/N \). Finally, to simplify the exposition, let us assume a uniform distribution of the quality of the products, that is, \( S(N) = \{u \in [\underline{u}, \bar{u}] : u = \underline{u} + k \Delta(N), k = 0, 1, \ldots, N\} \). This last assumption is not particularly restrictive as long as the mesh of \( S(N) \) goes to 0 as \( N \to \infty \).

With a little abuse of notation, we denote \( W(N) \) as the revenue rate given there are \( N \) products in the assortment. With no inventory constraint, one can show that the optimization problem for a fixed \( N \) reduces to maximizing the expected revenue rate which is given by

\[
W(N) = \max_{p \in \mathcal{A}(I_S)} \sum_{u \in S(N)} p(u) \left[ F(p^*(u), u - \alpha p(u)) - F(p(u), u - \alpha p(u)) \right],
\]

where \( p(u) \) is the price of product \( u \in S(N) \) and \( p^*(u) \triangleq p(u + \Delta(N)) \) for \( u < \underline{u} \) and \( p^*(\bar{u}) \triangleq \infty \).

Proposition 1 leads to \( 0 \leq p^*(u) - p(u) \leq \Delta(N)/\alpha \) for \( u \leq \bar{u} \). Using the continuity of \( F \), it follows that for \( \underline{u} \leq u < \bar{u} \)

\[
F(p^*(u), u - \alpha p(u)) - F(p(u), u - \alpha p(u)) = F_p(p(u), u - \alpha p(u))(p^*(u) - p(u)) + O(\Delta(N)).
\]

Hence, passing to the limit as \( N \to \infty \), we get

\[
W(\infty) = \sup_{p(u)} \int_{\underline{u}}^{\bar{u}} p(u) F_p(p(u), u - \alpha p(u)) \, dp(u) + p(\bar{u}) \left[ F(\infty, \bar{u} - \alpha p(\bar{u})) - F(p(\bar{u}), \bar{u} - \alpha p(\bar{u})) \right]
\]

subject to \( 0 \leq dp(u) \leq \frac{1}{\alpha} \, du \). (13)

(14)

This is a calculus of variation problem on the unknown function \( p(u) \) for \( u \in [\underline{u}, \bar{u}] \). The following proposition shows that the optimal price is a piece-wise linear function of the quality.

**Proposition 4** Let \( \bar{p} \) be an optimal solution to

\[
W(\infty) = \max_{p \geq 0} \left\{ \bar{p} F(\infty, \bar{u} - \alpha \bar{p}) - \bar{p} F(p, \bar{u} - \alpha \bar{p}) - \int_{p}^{\bar{p}} F(p, \bar{u} - \alpha \bar{p}) \, dp \right\} \quad \text{subject to} \quad p = \left[ \bar{p} - \frac{\bar{u} - u}{\alpha} \right]^+.
\]
Then, an optimal pricing strategy for the optimization problem (13) - (14) is given by

\[ p(u) = \left[ \bar{p} - \frac{\bar{u} - u}{\alpha} \right]^+ \quad u \in [u, \bar{u}], \tag{15} \]

A few remarks about this result are in order.

1. Depending on the value of \( p \) it is possible that some of the low quality products end up having no demand at all. Indeed, if the range of qualities offered in the assortment is sufficiently large (in the sense that \( \alpha \bar{p} < \bar{u} - u \)) then it would be optimal to set the price equal to zero for all products \( u \) with intrinsic utility \( u \leq \bar{u} - \alpha \bar{p} \). Hence, the demand for those products with \( u < \bar{u} - \alpha \bar{p} \) is effectively “shut down” since products in this group are dominated by product \( \bar{u} - \alpha \bar{p} \) that has the same price of 0 and offers a higher utility. From a practical standpoint, this result suggests that it is not necessarily optimal to seek expanding the breadth (or, the range of products) in the retailer’s assortment (specially to low quality products with utility \( u < \bar{u} - \alpha \bar{p} \)). Note that the previous analysis does not include procurement costs, which we have assumed are sunk.

2. At optimality, all products \( u \) with positive demand satisfy \( u - \alpha p(u) = \bar{u} - \alpha \bar{p} \). Recall that \( U = u - \alpha p \) measures the net utility received by a consumer that buys product \( u \) and pays \( p \) for it. Hence, in an optimal solution, buyers are indifferent among the subset of products they can buy since they get the same net utility independent of their wealth \( w \).

3. We next argue that in the limit as \( N \to \infty \), the inventory constraint drops out of the retailer’s pricing problem. To verify this claim note that for a finite \( N \) the pricing rule (15) implies that the probability that an arriving buyer requests product \( u \in S(N) \) is equal to

\[ F(p^*(u), u - \alpha p(u)) - F(p(u), u - \alpha p(u)) \leq \frac{\Delta(N)}{\alpha} + O(\Delta(N)), \quad u \in [u, \bar{u}] \]

Hence, the probability of a positive demand for product \( u \in [u, \bar{u}] \) goes to 0 as \( N \to \infty \). Therefore, as \( N \to \infty \), even with limited supply, the inventory constraint will not be binding, and the solution to the limited-supply case converges to that of the unlimited-supply case.

4. Figure 5(a) plots the value of an optimal pricing policy \( \{p_i = p(u_i) : i = 1, \ldots, N\} \) for different values of \( N \) under a bivariate exponential distribution for \((w, u_0)\). Consistent with our asymptotic analysis, optimal prices converge to the solution in equation (15) as \( N \to \infty \). This limiting policy corresponds to the piecewise linear function in solid line. Note that for small values of \( u \), the optimal price \( p(u) \) decreases with \( N \). The opposite is true when \( u \) is high.

The effect of the number of choices over total revenues is shown in Figure 5(b). As expected,
Product: $u_i$
Price: $p_i = p(u_i)$

![Figure 5](image)

Figure 5: (a) Pricing policies as a function of the number of substitute products $N = 2, 5, 10, 20, 50, 100$. The solid line is the limiting pricing rule in equation (15). (b) Expected Revenue as a function of $N$.

Data: $\bar{u} = 10$, $\underline{u} = 1$, $\alpha = 1$ and $F(p, u) = (1 - e^{-p/\theta_1}) (1 - e^{-u/\theta_2}) (1 + \delta e^{-p/\theta_1 - u/\theta_2})$, $\delta = 0.9$, $\theta_1 = 5$, $\theta_2 = 6$.

revenues reach an asymptotical limit as the number of products grows infinitely large. This limit is given by $W(\infty)$ in Proposition 4. As we can see from the figure, this limit is approached quite rapidly. Indeed, the expected revenue generated using two products is 64% of the maximum $W(\infty)$. This number goes up to 93% when the assortment includes ten products. This result is consistent with actual retailing practices, where it is quite uncommon to observe more than ten substitute brands compete simultaneously in a certain category of products.

Remark 4 above raises the question of how many products should be offered in one assortment. The following result gives us some simple bounds on the ratio between the expected revenue if the assortment includes only two product and the expected revenue if $N = \infty$.

**Proposition 5** Suppose the retailer is restricted to select two products from the entire menu $S(\infty) = [\underline{u}, \bar{u}]$. Let $W(2)$ be the optimal expected revenue in this case (assuming unlimited supply of the two products). Let $\bar{p}$ and $W(\infty)$ the optimal price and revenue (as defined in Proposition 4) when the retailer has no restrictions on the number of products that he can select. Then,

$$
\frac{F(\infty, \bar{u} - \alpha \bar{p}) - F(\bar{p}, \bar{u} - \alpha \bar{p})}{F(\infty, \bar{u} - \alpha \bar{p}) - F(p, \bar{u} - \alpha \bar{p})} \leq \frac{W(2)}{W(\infty)} \leq 1, \quad \text{where} \quad p = \left[ \bar{p} - \frac{\bar{u} - u}{\alpha} \right]^+.
$$

4 Admissible Pricing Policies and Problem Formulation

In this section we formulate the seller’s dynamic pricing problem when there is limited inventory constraint. Recall that under the MP2-1 model consumers are segmented according to their type $(\omega, u_0)$, where $\omega$ is the budget and $u_0$ is the (intrinsic) reservation utility, respectively. Each product $i$ has an intrinsic utility $u_i$ common to all buyers. The retailer’s objective is to optimally control the price vector $p_S(t) = (p_1(t), \ldots, p_N(t))$, over the selling horizon of length $T$, so as to maximize expected revenues given a vector of initial stocks $I_S(0) = (I_1(0), \ldots, I_N(0))$. 

24
The first step towards a mathematical formulation of the pricing problem is to extend the MP2-1 model to incorporate the possibility of inventory stock-outs and the corresponding demand substitution. One possible way is to dynamically adjust the set of products $S$ to include only those products with positive inventory. Another alternative, which is the one that we adopt here, is to keep the set $S$ fixed over time but modifying the pricing policy in such a way that $q_i(p_S(t)) = 0$ if $I_i(t) = 0$. This approach of capturing inventory-driven substitution using price-driven substitution is common in the Revenue Management literature, see for example Gallego and van Ryzin (1997).

Under the choice model considered, we can “shut down” the demand for product $i$ at any time $t$ if we set its price equal to the next best alternative, that is, $p_i(t) = p_{i-1}(t)$ (see equation (2)). This follows from the fact that the utility of product $i-1$ is higher than the utility of product $i$ and so rational buyer will purchase product $i-1$ instead of product $i$ when their prices are equal. In the case of product 1, recall that we have defined $p_0 = \infty$. In general, we will model this shut-down condition by requiring that $(p_i - p_{i-1}) I_i(0) = 0$ for all $i = 1, \ldots, N$, where $I(A)$ denotes the indicator function of the event $A$.

Our first result shows that, with limited supply, the optimal price of product $i$ is increases in the utility level $u_i$, which is a generalized result of Proposition 1.

**Proposition 6** Consider the problem of pricing $N$ products with limited inventory. Suppose that customers behave according to the MP2-1 model. Suppose also that the products are ordered such that their utilities satisfy $u_1 \geq u_2 \geq \cdots \geq u_N$. Then the optimal price vector $p_S^*(t)$ belongs to $P_S(t) \triangleq \{p_S(t) : p_{i+1}(t) \leq p_i(t) \leq p_{i+1}(t) + \psi_{i+1}, \text{ for all } i = 1, \ldots, N-1\}$ for $0 \leq t \leq T$, where $\psi_i = u_{i+1} - u_i$ for all $i = 2, 3, \ldots, N$.

According to this result, an optimal pricing function, $p^*(u)$ mapping products’ utility to prices, is an increasing function of $u$ and has a slope bounded above by $1/\alpha$. Hence, as the the buyer’s marginal utility to residual budget $\alpha$ increases, it becomes harder for seller to price discriminate based on the quality of the products.

Based on our previous discussion, we say that the price process $p_S(t)$ is admissible if the following two conditions are satisfied for all $t \in [0, T]$:

(i) $(p_{i-1}(t) - p_i(t)) I_i(t) = 0$ for all $i = 1, \ldots, N$

and (ii) $p_S(t) \in P_S$. The set of admissible price processes will be denoted by $A(I_S)$. With this definition, we can write the retailer’s optimization process as the following stochastic control problem.

$$\max_{p_S \in A(I_S)} -E \left[ \int_0^T p_S(t) \cdot dI_S(t) \right]$$

subject to $I_i(t) = I_i(0) - D_i \left( \lambda \int_0^t q_i(p_S(\tau)) d\tau \right)$ for all $i \in S$, 

where $\{D_i(\lambda t) : i \in S\}$ is a set of independent Poisson processes of rate $\lambda$.

As usual, a solution to (16)-(17) can be searched using dynamic programming. For this, let us
introduce the value function \( V(t, I_S) \) representing the optimal expected revenue if the inventory position is \( I_S \) and the remaining time is \( t \). Note that in this definition of \( V(t, I_S) \) time runs backward. Under the assumption that \( V(t, I_S) \) is differentiable in \( t \), the Hamilton-Jacobi-Bellman (HJB) optimality condition is given by the difference-differential equation (e.g., chapter VII in Bremaud (1981))

\[
\frac{\partial V(t, I_S)}{\partial t} = \max_{p_S \in \mathcal{A}(I_S)} \left\{ \lambda \sum_{i=1}^{N} q_i(p_{i-1}, p_i) [p_i + V(t, I_S - e_i) - V(t, I_S)] \right\},
\]

where \( e_i \) is the \( N \)-dimensional canonical vector having a one in the \( i \)th component and zero elsewhere. The boundary conditions are:

\[
V(T, I_S) = V(t, 0) = 0 \quad \text{for all } t \in [0, T] \quad \text{and} \quad I_S \in \mathbb{Z}^N_+.
\]

As in most of the dynamic pricing problems, a closed-form derivation of an optimal solution to (16)-(17) is not available. The multi-dimensional nature of the problem adds more difficulty to solving the closed-form solution. However, we are able to make a few qualitative statement about the expected revenue and optimal pricing policy based on the following analysis.

First, we define the Fenchel-Legendre operator \( \Phi(\cdot) : \mathbb{R}^N \times \mathbb{Z}^N_+ \rightarrow \mathbb{R} \) associated to the instantaneous revenue rate as follows. For every \( Z = (z_i) \in \mathbb{R}^N \)

\[
\Phi(Z, I_S) \triangleq \max_{p_S \in \mathcal{A}(I_S)} \left\{ \lambda \sum_{i=1}^{N} q_i(p_{i-1}, p_i) [p_i - z_i] \right\}.
\]

Note that \( \Phi(Z, I_S) \) depends on the inventory \( I_S \) through the set of admissible prices \( \mathcal{A}(I_S) \). This dependence, however, is rather weak since the only information that matters about \( I_S \) to compute \( \Phi(Z) \) is the set of products with zero inventory. This follows from the requirement that \( p_i = p_{i-1} \) if \( I_i = 0 \). The following is a well-known result in convex analysis (e.g. see Rockafeller (1997)), and therefore the proof is omitted.

**Lemma 1** The Fenchel-Legendre operator \( \Phi(Z, I_S) \) is nonnegative, decreasing and convex in \( Z \) for every \( I_S \geq 0 \).

To compute \( \Phi(Z) \) we must solve the multidimensional nonlinear optimization in (20). In our setting, we can solve this optimization efficiently due to special form of the choice model MP2-1. Recall that MP2-1 defines a unique ranking of the products in \( S \) through the intrinsic utilities \( u_i \). This turns out to be a very useful property that we can exploit to solve the maximization problem in (18) using dynamic programming techniques. Indeed, for the pair of \( N \)-dimensional vectors
(Z, I_S), let us define the auxiliary value function \( W_k(p_{k-1}; Z, I_S) \) as follows

\[
W_k(p_{k-1}; Z, I_S) = \max_{p_k;...;p_N} \mathbb{E} \left[ \sum_{i=k}^{N} q_i(p_{i-1}, p_i) [p_i - z_i] \right]
\]

subject to \( p_{i-1} - \psi_i \leq p_i \leq p_{i-1} \) and \( (p_{i-1} - p_i) \mathbb{I}(I_i = 0) = 0, \) \( i = k, \ldots, N. \)

The two constraints follow from the requirement \( p_S \in A(I_S). \) Note that \( W_k(p_{k-1}; Z, I_S) \) is equal to the maximum instantaneous expected revenue rate obtained from products \( k, k+1, \ldots, N \) given \( p_{k-1}, \) the posted price of product \( k - 1. \) The resulting Bellman equation for \( W_k(p_{k-1}; Z, I_S) \) is given by

\[
W_k(p_{k-1}; Z, I_S) = \begin{cases} 
W_{k+1}(p_{k-1}; Z, I_S) & \text{if } I_i = 0 \\
\max_{p_{k-1} - \psi_k \leq p_k \leq p_{k-1}} \left\{ q_k(p_{k-1}, p_k) [p_k - z_k] + W_{k+1}(p_k; Z, I_S) \right\} & \text{if } I_i > 0
\end{cases}
\]

with boundary conditions,

\[
W_k(0; Z, I_S) = 0, \text{ for all } k \in S \quad \text{and} \quad W_{N+1}(p; Z, I_S) = 0, \text{ for all } p, Z, I_S \geq 0.
\]

Note that \( \Phi(Z, I_S) = W_1(\infty; Z, I_S). \) The following proposition is useful for characterizing the optimal solution to the dynamic program in (21).

**Lemma 2** Assume \( F \) satisfies the conditions in Assumption [1]. Then, for all \( (Z, I_S) \geq 0 \) the value function \( W_k(p; Z, I_S) \) is independent of \( z_i \) \( (i = 1, \ldots, k - 1), \) decreasing in \( z_j \) \( (j = k, \ldots, N) \) and increasing in \( p. \) In addition, suppose \( I_k > 0 \) then the optimal price in stage \( k \)

\[
p_k^*(p; Z, I_S) \triangleq \arg\max_{p - \psi_k \leq p_k \leq p} \left\{ q_k(p, p_k) [p_k - z_k] + W_{k+1}(p_k; Z, I_S) \right\}
\]

is an increasing function of \( p \) and \( z_k. \)

We can now use Lemma [1] and Lemma [2] to prove the following general result about the solution to the dynamic pricing problem [16]. Let us first define \( \Delta_i V(t, I_S) \triangleq V(t, I_S) - V(t, I_S - e_i) \) to be the marginal value of an additional unit of product \( i \) if the current inventory is \( I_S \) and the remaining time is \( t. \) This definition assumes that \( V(t, I_S - e_i) = 0 \) if \( I_i = 0. \)

**Theorem 1** Suppose \( V(t, I_S) \) is a solution to the HJB equation in (18) that is continuously differentiable in \( t. \) Then,

a) \( V(t, I_S) \) is decreasing in \( t \) and increasing in each component of \( I_S. \)

b) \( \Delta_i V(t, I_S) \) is increasing in \( t \) and decreasing in each component of \( I_S. \)

c) Under the conditions in Proposition [3] the optimal price strategy \( p^*(t, I_S) \) is increasing in \( t \) and decreasing in every component of \( I_S. \)
Theorem 1 as well as Proposition 6 have exhibited the following properties of the optimal prices: (1) quality monotonicity: a product with higher quality should be priced higher than a low-quality product; (2) time monotonicity: as the end of selling approaches, all products should be priced lower; (3) inventory monotonicity: the price of each product decreases in the inventory level of each product in the assortment. The similar results have been shown in Ackay et al. (2010), where they also analyze vertically differentiated products but without customer’s budget constraints and market segmentation.

From an analytical perspective, the main difficulties for solving problem (16)-(17) are driven by three main factors: (a) the inventory constraints in (17), (b) the stochastic nature of the problem, and (c) the spill-over effect of stock-outs (or inventory-driven substitution), that is, if we run out of stock for product $i$ then some demand for $i$ will shift to other products. Factor (a) and (b) are also crucial to the continuous-time pricing problem for only one product, since the firm needs to dynamically adjust the pricing policy to maximize her revenue from the limited inventory. With multiple products, customer’s substitution behavior as described in (c) will only make the problem even more nontrivial.

For the above reasons, approximated solutions are needed based on different types of asymptotic analysis. In Section 3, we have considered the special case of unlimited supply where $I_S(0) \uparrow \infty$. This allows us to ignore factor (a) and (c) which associate with the limited inventory. As we have seen, this approximation simplifies considerably the analysis as the optimal solution becomes not only inventory independent but also time independent.

Our next approximation deals with factor (b). In Section 5 we consider the deterministic counterpart of problem (16)-(17). We will argue that this deterministic problem can be viewed as a limiting situation in which both the vector of initial inventory $I_S(0)$ and the demand rate $\lambda$ increase proportionally large. Thus, this deterministic formulation can be viewed as a good approximation for large retail operations.

In what follows, we will also limit our analysis to the case where the value of residual money is equal to zero, i.e. $\alpha = 0$. The pricing problem under the more general non-zero $\alpha$ is left as an extension for future research. A natural exploration avenue is the utilization of Taylor expansion to approximate $q_i(p_S)$. As a particular case, the uniform distribution (for which $q_i(p_S)$ presents a closed form expression) should be analyzed to obtain preliminary insights.

5 Deterministic Approximation to the Finite Inventory Case

In this section we propose a deterministic approximation for problem (16)-(17), where demand is modeled in a fluid-like (continuous) and time-homogeneous way. This approximation will simplify the path-dependent nature of the pricing problem, allowing a more tractable analytical formulation.
We will see that this deterministic continuous approximation is asymptotically optimal as the volume of expected sales and initial inventory grow proportionally large. We will first provide a line-search algorithm to compute the optimal solution, and then we will characterize the sufficient and necessary conditions on the existence of the optimal solution. We then apply the first-order approximation on the optimality condition for the pricing policy to generate a pricing heuristic as well as the bounds of the price ratios of products with different qualities. The numerical experiments show that the revenue generated from the heuristic pricing policy is quite satisfactory compared with the maximum revenue.

5.1 Model description

Consider a sequence of instances of problem (16)-(17) parameterized by \( n \in \mathbb{Z}^+ \). For the \( n \)th instance, let us denote by \( I^n_S(0) \) and \( \lambda^n \) the vector of initial inventory and demand rate, respectively. All other parameters are kept fixed independent of \( n \). In the limiting regime that we consider, we let both \( I^n_S(0) \) and \( \lambda^n \) grow proportionally large. In other words, we consider those regimes that approximate the operations of a large retailer. Specifically, we define

\[
I^n_S(0) = nI_S(0) \quad \text{and} \quad \lambda^n = n\lambda,
\]

where \( I_S(0) \) and \( \lambda \) are constants. For the \( n \)th instance, the retailer’s optimization problem (16)-(17) becomes

\[
V^n \triangleq \max_{p_S \in A} -\mathbb{E} \left[ \int_0^T p_S(t) \cdot dI^n_S(t) \right]
\]

subject to \( I^n_i(t) = I^n_i(0) - D_i \left( \lambda^n \int_0^t q_i(p_S(\tau)) \, d\tau \right) \), for all \( i \in S \).

We note that the set \( A \) of admissible pricing policies remains independent of \( n \). Our next step is to consider a normalized version of the optimization problem above. To this end, let us introduce the following scaled quantities:

\[
\bar{V}^n \triangleq \frac{V^n}{n} \quad \text{and} \quad \bar{I}^n_i(t) \triangleq \frac{I^n_i(t)}{n}, \quad \text{for all } i \in S.
\]

Combining these definitions and the asymptotic regime given by condition (22), we obtain the following equivalent formulation

\[
\bar{V}^n \triangleq \max_{p_S \in A} -\mathbb{E} \left[ \int_0^T p_S(t) \cdot d\bar{I}^n_S(t) \right]
\]

subject to \( \bar{I}^n_i(t) = I_i(0) - \frac{1}{n} D_i \left( n\lambda \int_0^t q_i(p_S(\tau)) \, d\tau \right) \), for all \( i \in S \).

\[\text{†} \]

A more general definition of our asymptotic regime given by \( \lim_{n \to \infty} I^n_S(0) = I_S(0) \) and \( \lim_{n \to \infty} \frac{\lambda^n}{n} = \lambda \) is possible, but for ease of exposition we restrict ourselves to the special case in (22).
For any pricing policy \( p_S(t) \in A \) and any product \( i \in S \), the demand intensity process
\[
\lambda \int_0^t q_i(p_S(\tau)) \, d\tau
\]
is continuous and uniformly bounded in \([0, T]\). Therefore, in the limit as \( n \uparrow \infty \) the scaled inventory process \( \bar{I}_i^n(t) \) converges (almost surely and uniformly over a compact set) to a process \( I_i(t) \) such that
\[
\lim_{n \to \infty} \bar{I}_i^n(t) \overset{a.s.}{=} I_i(t) \text{ u.o.c., where } I_i(t) = I_i(0) - \lambda \int_0^t q_i(p_S(\tau)) \, d\tau.
\]
We do not attempt a formal proof of this convergence as it goes beyond the scope of this paper. For further details on this type of convergence and limiting regimes, the interested reader is referred to Kurtz (1978), Mandelbaum and Pats (1995), and reference therein.

Under this asymptotic regime, the retailer’s pricing problem (16)-(17) reduces to the following deterministic continuous time control problem.

\[
V^{\text{det}}(I_S(0)) \triangleq \max_{p_S \in A} \lambda \int_0^T p_S(t) \cdot q_S(p_S(t)) \, dt
\]
subject to \( I_i(t) = I_i(0) - \lambda \int_0^t q_i(p_S(\tau)) \, d\tau \) for all \( i \in S \).

Note that the optimization problem is autonomous in the sense that the demand rate is constant and the set \( A \) and the functions \( \{q_i(p_S) : i = 1, \ldots, N\} \) are independent of the calendar time \( t \). Therefore, we can search for an optimal policy within the family of pricing policies that are constant over time, that is, solving the finite dimensional optimization problem

\[
V^{\text{det}}(I_S(0)) \triangleq \max_{p_S \in A} \lambda T \sum_{i=1}^N p_i q_i(p_S)
\]
subject to \( \lambda T q_i(p_S) \leq I_i(0) \) for \( i = 1, \ldots, N \).

### 5.2 Optimal Pricing Policy

#### 5.2.1 Algorithm for Computing Optimal Pricing Policy

Again, rather than solving the dynamic program, we can use a much simpler line-search algorithm to compute the optimal solution. In fact, the Karush-Kuhn-Tucker (KKT) optimality conditions for problem (26) are (e.g., Chapter 4 in Bazaraa et al. (1993))

\[
0 = F(p_{i-1}, u_i) - F(p_i, u_i) - (p_i - \nu_i) F_p(p_i, u_i) + (p_{i+1} - \nu_{i+1}) F_p(p_i, u_{i+1}) \quad \text{for all } i = 1, \ldots, N
\]
\[
0 \leq I_i(0) - \lambda T \left[ F(p_{i-1}, u_i) - F(p_i, u_i) \right], \quad \text{for all } i = 1, \ldots, N
\]
\[
0 = \nu_i \left( I_i(0) - \lambda T \left[ F(p_{i-1}, u_i) - F(p_i, u_i) \right] \right), \quad \text{for all } i = 1, \ldots, N
\]
\[
0 \leq \nu_i \quad \text{for all } i = 1, \ldots, N
\]
where \( \nu_i \) is the Lagrangian multiplier for the \( i \)th product inventory constraint. The following algorithm characterizes a line-search procedure that simultaneously computes a vector of prices and multipliers that solve the KKT conditions above.

**Limited Inventory Algorithm:**

**Step 1:** Set \( p_{N+1} = \nu_{N+1} = 0 \) and fix \( p_N = \bar{p} \).

**Step 2:** For \( i = N, \ldots, 2 \) compute \( p_{i-1} \) as a function of \( \bar{p} \) as follows. Given \( p_i, p_{i+1} \) and \( \nu_{i+1} \), compute

\[
\begin{align*}
\zeta_i & \triangleq \min \left\{ \frac{I_i(0)}{\lambda T}, p_i F_p(p_i, u_i) - (p_{i+1} - \nu_{i+1}) F_p(p_i, u_{i+1}) \right\} \quad \text{and} \\
\hat{p} & \triangleq F^{-1}(F(p_i, u_i) + p_i F_p(p_i, u_i) - (p_{i+1} - \nu_{i+1}) F_p(p_i, u_{i+1}), u_i)
\end{align*}
\]

Then,

\[
p_{i-1} = F^{-1}(F(p_i, u_i) + \zeta_i, u_i) \quad \text{and} \quad \nu_i = \frac{F(\hat{p}, u_i) - F(p_{i-1}, u_i)}{F_p(p_i, u_i)}.
\]

**Step 3:** Optimality check: if there exists an \( \nu_1 \geq 0 \) such that

\[
F(p_0, u_1) - F(p_1, u_1) - (p_1 - \nu_1) F_p(p_1, u_1) + (p_2 - \nu_2) F_p(p_1, u_2) = 0,
\]

\[
\lambda T[F(p_0, u_1) - F(p_1, u_1)] \leq I_1(0), \quad \text{and} \quad \nu_1 \left( \lambda T[F(p_0, u_1) - F(p_1, u_1)] - I_1(0) \right) = 0,
\]

then the sequences \( \{p_1, \ldots, p_N\} \) and \( \{\nu_1, \ldots, \nu_N\} \) jointly satisfy the KKT conditions and stop. Otherwise, go to Step 1, change the value of \( \bar{p} \) and iterate. \( \square \)

A couple of observations about this algorithm are in order. First of all, by construction in Step 2 \( \hat{p} \geq p_{i-1} \) which guarantees that \( \nu_i \geq 0 \). Also from Step 2 note that

\[
F(p_{i-1}, u_i) - F(p_i, u_i) = \zeta_i \geq \frac{I_i(0)}{\lambda T}
\]

which guarantees that the inventory constraint for product \( i = 2, \ldots, N \) is satisfied. In addition, if the inequality is strict it follows that \( \hat{p} = p_{i-1} \), that is, \( \nu_i = 0 \) and so the complementary slackness condition is satisfied for \( i = 2, \ldots, N \). If \( \zeta = \frac{I_i(0)}{\lambda T} \), then the inventory constraint is binding and the complementary slackness condition is also satisfied for \( i = 2, \ldots, N \).

From Step 2 and the definitions of \( \hat{p}, \zeta_i, p_{i-1} \) and \( \nu_i \) the reader can easily verify that the first KKT optimality condition is also satisfied for \( i = 2, \ldots, N \). Finally, the optimality check in Step 3 guarantees that the KKT conditions are also satisfied for \( i = 1 \). In summary, if the algorithm is able to find a solution, then this solution satisfies the KKT conditions in (27).

The question now is whether there exists a solution to the KKT conditions. The following result provides necessary and sufficient conditions for this to happen. We recall from Section 3 the
definitions of $L$ and $U$.

$$L(p, u_1, u_2) \triangleq F(p, u_1) + p\left( F_p(p, u_1) - F_p(p, u_2) \right)$$ and $$U(p, u_1) \triangleq F(p, u_1) + p F_p(p, u_1).$$

**Proposition 7** A necessary condition for the existence of a solution is that there is a $\tilde{p}$ such that $$U(\tilde{p}, u_1) = F(p_0, u_1).$$

On the other hand, suppose that $L(p, u_1, u_2)$ is unimodal. Then, a sufficient condition for the existence of a solution to the KKT conditions is that there exists a price $\hat{p}$ that solves $$L(\hat{p}, u_1, u_2) = F(p_0, u_1).$$

The previous algorithm provides us with the insights about the effects that a limited inventory has on an optimal pricing strategy. From step 2, we see that as the inventory of product $i$ decreases the prices of product $i$ and $i - 1$ get closer. In the limit, as $I_i(0)$ goes to zero, the price of product $i$ converges to the price of product $i - 1$ ($p_i \uparrow p_{i-1}$). The intuition is that products with small inventory have a high chance of stocking out during the selling season. In order to mitigate this effect, the seller raises the price of these products close to the next best alternative.

### 5.2.2 Numerical study

We next present a few numerical examples that highlight the effects of the inventory constraints. For this purpose, just as we did in Section 3, we consider an average arrival of 100 customers over the selling horizon ($D = 100$). Customers’ type is represented by a bivariate normal distribution with mean $(\mu_w, \mu_u) = (1, 1)$, variance $(\sigma_w^2, \sigma_u^2) = (0.5, 0.4)$, and coefficient of correlation $\rho = 0.5$. The quality of products is assumed to be evenly distributed between 0.5 and 3.

Figure 6 presents optimal revenues as a function of the initial stock of a certain product. In this numerical exercise we consider a family of 5 substitute products, where all products, with the exception of product 3, have unlimited supply.

The curve that appears in this figure has the expected form. It presents an increasing monotonicity with decreasing marginal increments. The maximum revenue is limited by the optimal non-limited selling amount (in this particular case around 20 units). It is interesting to observe that the first 15 units are responsible for more than 90% of the potential revenue attributable to product 3.

Figure 7 studies the effect over pricing policies, and selling amounts, generated by the presence of certain products with limited inventory. We consider a family of 10 substitute products, where products 3, 6, and 9 have a limited stock with only one unit of inventory (the rest of the products have unlimited supply).

As shown in Figure 7(a), the limited inventory price curve follows a stepwise form. Products
with limited inventory present a very small price drop from the price with a higher quality level, so as to match the demand with the scarce available supply. Products with immediately higher quality than the limited ones, have prices slightly above those of their restricted neighbors, so as to satisfy part of the unsatisfied demand due to stock-outs.

Figure 6(b) compares the optimal selling amounts of the limited and unlimited supply settings. This plot shows an increment on the selling quantities of all the products, except for those with limited inventory. This behavior is rather intuitive; when a certain product is limited, part of the customers’ demand for this product will be absorbed by higher or lower quality products.

Similarly to Section 3.2, let us conclude this section discussing a first-order approximation for problem (26).

5.3 Pricing Heuristics using First-Order Taylor Approximation

In this section, we use a first-order Taylor expansion to approximate $p_i$. From step 2 in the Limited Inventory algorithm it follows that $p_{i-1} = F^{-1}(F(p_i, u_i) + \zeta_i, u_i)$. Using a first-order approximation of $F^{-1}(x, u_i)$ around $x = F(p_i, u_i)$ and the definition of $\zeta_i$ we get

$$p_{i-1} \approx p_i + \frac{\zeta_i}{F_p(p_i, u_i)} = p_i + \min \left\{ \frac{I_i(0)}{\lambda T F_p(p_i, u_i)}, p_i - (p_{i+1} - u_{i+1}) \frac{F_p(p_i, u_{i+1})}{F_p(p_i, u_i)} \right\}.$$ 

Like §3.2 we use the fact that $\frac{F_p(p_i, u_{i+1})}{F_p(p_i, u_i)} \in [0, 1]$ to get the following bounds for $p_{i-1}$.

$$p_i + \min\left\{ \frac{I_i(0)}{\lambda T F_p(p_i, u_i)}, p_i - p_{i+1} \right\} \leq p_{i-1} \leq p_i + \min\left\{ \frac{I_i(0)}{\lambda T F_p(p_i, u_i)}, p_i \right\}. \tag{28}$$

A couple of observation about the bounds in (28) are in order. First note that if the inventory of
product $i$ is zero then $p_i = p_{i-1}$, that is, the bounds are tight. On the other hand, if the inventory of product $i$ is large enough then the bounds in (28) coincide with those obtained in section 3.2.

In what follows, we present some numerical experiments that show the quality of this approximation, and its adequacy in the implementation of a simple pricing methodology. In these experiments we consider a family of 10 substitute products, and assume all of these have unlimited inventory levels, except for products 3, 6 and 9, which have a single unit only. For the ease of notation, we define

$$p_i = p_{i+1} + \min\{\gamma_i + 1, p_{i+1} - p_{i+2}\}$$

and

$$\bar{p}_i = p_{i+1} + \min\{\gamma_i + 1, p_{i+1}\},$$

where $\gamma_i = \frac{I_i(0)}{\lambda T F(p_i, \mu_i)}$. Note that $\gamma_i$ can be considered as the clearing price of product $i$.

Table 3 analyzes the approximation quality of the bounds of optimal prices under the same two customer segmentation schemes presented in 3.2: a bivariate weibull distribution and a bivariate normal distribution. The first three columns of the table characterize the product according to its quality (utility) and inventory level. The following eight columns present the optimal selling level, the optimal price ($p_i^*$), and the lower and upper bounds ($p_i$ and $\bar{p}_i$), for each of the ten products under both segmentation settings.

The results shown in this table indicate that the quality of the approximation presented in (28) is satisfactory under both studied distributions. Except for products 1 and 9 in the bivariate normal case, optimal prices are contained within the approximated bounds. This approximation will be useful, however, in the way it can be adequately implemented in a pricing methodology. To analyze this issue, we proceed in a similar way as we did in Section 3.2. First, let us define the approximated price for product $i$ as $\tilde{p}_i(a) \triangleq a p_i + (1 - a) \bar{p}_i$, where $a \in [0, 1]$ is a fixed constant.
To obtain the set of approximated prices, \( \tilde{p}_i \), we require a fixed value for \( \tilde{p}_N \). To do so, a possible approach is to consider the solution of the unlimited case with a uniform distribution. Once \( \tilde{p}_N \) has been established, it is possible to compute \( \tilde{p}_i \) for \( i = N - 1, \ldots, 1 \). In Table 4, we compare optimal revenues with those obtained applying the pricing strategy just described. Let \( R \) be the ratio between the revenues generated by using \( \tilde{p}_S \) and the optimal revenue.

Just as we did in the experiments of the previous section, we assume customers are characterized by a bivariate normal distribution with \( \mu_w = \mu_{u0} = 1 \) and \( \sigma_w = \sigma_{u0} = 1 \). We analyze three different levels of correlation (\( \rho = 0.1, 0.5, 0.9 \)) for a family of 10 substitute products, whose quality (utility) is randomly distributed over \([\mu_{u0} - \sigma_{u0}, \mu_{u0} + \sigma_{u0}]\). For each of these three values of \( \rho \), we consider two different \( a \)'s, \( a = 1.0 \) and \( a = 0.7 \). For each of the studied combinations, a set of 100 random instances of product quality were generated to compute the mean and standard deviation of \( R \) (\( R_{\text{mean}} \) and \( R_{\text{std}} \), respectively).

As mentioned earlier, the price of product \( N \) has to be assigned exogenously. To do so, we use an approximation which is calculated using the unlimited case with the following bivariate uniform distribution

\[
\mathbb{P}[w \leq p, u_0 \leq u] = \left( \frac{p - (\mu_w - \sigma_w)}{2\sigma_w} \right) \left( \frac{u - (\mu_{u0} - \sigma_{u0})}{2\sigma_{u0}} \right).
\]

The price of product \( N \) generated in this way is denominated \( p_{N,\text{unif}}^N \).

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( a = 1.0 )</th>
<th>( a = 0.7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_{\text{mean}} )</td>
<td>( R_{\text{std}} )</td>
<td>( R_{\text{mean}} )</td>
</tr>
<tr>
<td>0.1</td>
<td>.7667</td>
<td>.0214</td>
</tr>
<tr>
<td>0.5</td>
<td>.8030</td>
<td>.0249</td>
</tr>
<tr>
<td>0.9</td>
<td>.8418</td>
<td>.0318</td>
</tr>
</tbody>
</table>

Table 4: Revenues applying the approximation \( \tilde{p}_S \) versus optimal revenues in the limited inventory case (only products 3, 6 and 9 are limited with a single unit).
The results shown in Table 4, though not as good as those presented in the previous section, are still quite promising (the lowest $r_{mean}$ is above 0.75, while the highest is almost 0.88). It is also possible to observe, that in this limited case, the value of $a$ is not as important as in the unlimited case (at least, under the studied conditions). This last result is quite reasonable since, as shown in Table 3, neither of both bounds is clearly more accurate than the other. Considering these results, we believe the methodology presented here constitutes a very efficient and robust way to establish prices under a limited inventory setting, specially when the demand distribution is not known.

6 Conclusions

This paper has studied the optimal pricing problem for perishable products with demand substitution. We provided an original demand model, what we call the MP2-1 model, that allows for an adequate characterization of customers’ purchasing decisions under a price and inventory driven substitution environment. Some of the properties presented by this demand model, such as the ability of establishing a ranking among products, and its greedy resolution nature, overcome most of the limitations inherent to other commonly used models (e.g. MNL). Based on this demand model, we formulated the retailer’s pricing problem as a stochastic control problem, and identified three main factors that account for the complexity of its resolution: (i) the inventory constraints, (ii) the stochastic nature of the problem, and (iii) the presence of inventory driven substitution. To deal with these difficulties, we considered two different asymptotic approximations that permitted us the attainment of further insights.

Our first asymptotic approximation, the unlimited supply case, eliminates inventory and spillover effects. In this infinite inventory setting, we proposed an efficient line-search procedure to obtain optimal prices, and derived properties of optimal prices through a first-order Taylor approximation of $F^{-1}(p,u)$. These properties do not only establish a characterization of optimal prices, but can be used to define pricing policies in an efficient and distribution-free way, which constitutes an attractive methodology to support managerial decisions. The adequacy of these properties were tested through a set of computational experiments. The second asymptotic approximation consisted in modeling demand in a deterministic fluid-like way, which overcomes the effects of stock-outs and the stochastic nature of the problem. We showed that this deterministic continuous approximation is the limit of the original model when the volume of expected sales and initial inventories grow proportionally large. We presented a line-search procedure that allows for an efficient resolution of the resulting KKT optimality conditions. Finally, as in the unlimited case, we developed a distribution-free pricing methodology, and tested its adequacy with a set of computational experiments.

An interesting direction for future research is to study the problem of jointly optimizing pricing
and assortment policies. The optimal assortment problem with substitutable products, as indicated in Section 1, has received plenty of attention (e.g., van Ryzin and Mahajan (1999), Smith and Agrawal (2000), and Mahajan and van Ryzin (2001)). However, all these works have excluded pricing decisions in their analysis, assuming prices are determined exogenously. In this way, the problem of optimizing pricing and assortment decisions together under the MP2-1 model, is a natural extension of our work that could be explored. Another future direction is to allow the strategic consumer behavior into the choice model. Parlakturk (2012) has taken the first step and considered a firm that sells two vertically differentiated products to strategically forward-looking consumers over two periods. Liu and Zhang (2013) study dynamic pricing competition between two firms offering two vertically differentiated products to strategic consumers. We believe that research related to the strategic consumer behavior is still an under-developed area for future study. We expect to see more opportunities on dynamic pricing for products with correlated demands with forward-looking behavior.

References


Appendix

Proof of Proposition 1: For any given price vector \( p_S = \{p_1, p_2, ..., p_N\} \), we define \( S_e \subseteq S \) as the set of effective products. Roughly speaking, for each effective product \( j \), there exists a customer type \((w, u_0)\) which will buy the product \( j \). We now define \( S_e \) in the following more rigorous way.

Step 0: Set \( S_e = \emptyset, j = 0, \) and \( p_0 = \infty \).
Step 1: Let \( k \triangleq \min \arg \max_{i \in S, p_i < p_j} (u_i - \alpha p_i) \). Then \( S_e \leftarrow S_e \cup \{k\} \), and \( j \leftarrow k \).

Step 2: If \( \arg \max_{i \in S, p_i < p_j} (u_i - \alpha p_i) = \emptyset \), then stop; otherwise, go to step 1.

We then relabel the element in \( S_e \) as \( S_e = \{i_1, i_2, i_3, \ldots\} \) such that \( u_{i_q} - \alpha p_{i_q} \) is a decreasing sequence in \( q \) and \( p_{i_q} \) strictly decreases in \( q \) for \( q = 1, 2, 3, \ldots \). Then \( i_q \) must be an increasing sequence in \( q \).

It can be verified that no customer would purchase the product \( i \), if \( i \notin S_e \). For product \( i_q \in S_e \), it will be purchased by customers with type \((w, u_0) \in [p_{i_q}, p_{i_q-1}] \times [0, u_{i_q} - \alpha p_{i_q}] \).

Then, we construct the new price path \( p_S' \) as follows. We first denote \( j \triangleq \min \{S - S_e\} \). Then find \( q \) such that \( i_q < j < i_{q+1} \). Then we want to find \( p_j' \) such that

\[
p_j' \in (p_{i_{q+1}}, p_{i_q}) \quad \text{and} \quad u_j - \alpha p_j' \in (u_{i_{q+1}} - \alpha p_{i_{q+1}}, u_{i_q} - \alpha p_{i_q})
\]

For completeness, we let \( u_0 = \infty \) and \( u_{N+1} = p_{N+1} = 0 \). To show the existence of \( p_j' \), note that if \( p_j' \in (p_{i_{q+1}}, p_{i_q}) \), then \( u_j - \alpha p_j' \in (u_j - \alpha p_{i_q}, u_j - \alpha p_{i_{q+1}}) \). Since \( u_j - \alpha p_{i_q} < u_{i_q} - \alpha p_{i_q} \) and \( u_j - \alpha p_{i_{q+1}} > u_{i_{q+1}} - \alpha p_{i_{q+1}} \), the intersection of the two open intervals \( (u_{i_{q+1}} - \alpha p_{i_{q+1}}, u_{i_q} - \alpha p_{i_q}) \cap (u_j - \alpha p_{i_q}, u_j - \alpha p_{i_{q+1}}) \) must not be an empty set, which proves the existence of \( p_j' \). Now we calculate the difference in the revenue rate. Note that

\[
W(p_S') - W(p_S) = p_j' \left( F(p_{i_q}, u_j - \alpha p_j') - F(p_j', u_j - \alpha p_j') \right) - p_{i_{q+1}} \left( F(p_{i_q}, u_{i_{q+1}} - \alpha p_{i_{q+1}}) - F(p_j', u_{i_{q+1}} - \alpha p_{i_{q+1}}) \right)
\]

Since \( p_j' > p_{i_{q+1}} \) and \( F(p_{i_q}, u_j - \alpha p_j') - F(p_j', u_j - \alpha p_j') \geq F(p_{i_q}, u_{i_{q+1}} - \alpha p_{i_{q+1}}) - F(p_j', u_{i_{q+1}} - \alpha p_{i_{q+1}}) \), we can conclude that \( W(p_S') \geq W(p_S) \). Proceeding similarly, we could find a strictly decreasing price path in \( i \in S \) to achieve a higher revenue rate than \( W(p_S) \).

\[\square\]

Proof of Proposition 3.2: We first note that for \( \bar{p} = 0 \) we have that \( p_i(\bar{p}) = 0 \) and for \( \bar{p} \to \infty \) we have that \( p_i(\bar{p}) \to \infty \), for all \( i = 1, \ldots, N \). Thus, by the continuity of the distribution function \( F(p, u) \), the range of the mapping \( p_i(\bar{p}) \) is the entire \( \mathbb{R}_+ \). This observation implies that the issue of existence reduces to find a price \( \bar{p} \in \mathbb{R}_+ \) such that condition 8 is satisfied.

Note that \( p_S(\bar{p}) \in P_S \) and so we can bound the right-hand side in 8 as follows.

\[
L(p_1(\bar{p}), u_1, u_2) \leq F(p_1(\bar{p}), u_1) + p_1(\bar{p}) F_p(p_1(\bar{p}), u_1) - p_2(\bar{p}) F_p(p_1(\bar{p}), u_2) \leq U(p_1(\bar{p}), u_1).
\]

Since the range of \( p_1(\bar{p}) \) is \( \mathbb{R}_+ \), a sufficient condition for the existence of a solution to condition 8 is that there exists a price \( \hat{p} \in \mathbb{R}_+ \) such that \( L(\hat{p}, u_1, u_2) = F(p_0, u_1) \). On the other hand, a necessary condition for the existence of a solution is that there exists a \( \hat{p} \in \mathbb{R}_+ \) such that \( U(\hat{p}, u_1) = F(p_0, u_1) \).

Finally, we prove that the sequences \( \{p_i^{\min}\} \) and \( \{p_i^{\max}\} \) generated by 9 and 10 are effectively lower and upper bounds on the solutions to condition 5. Denote \( h_i(p_i) \) as \( h_i(p_i) \triangleq F(p_i, u_i) + \).
This completes the proof. □

PROOF OF PROPOSITION 3. The recursion is given by $p_{i-1} = 2p_i - p_{i+1} F_p(p_i, u_{i+1})$. By the definition of $F(p, u)$, it follows that $F_p(p_i, u_{i+1}) \in [0, 1]$. Therefore we obtain

$$2p_i - p_{i+1} \leq p_{i-1} \leq 2p_i \quad (i = 1, \ldots, N).$$

(29)

The inequality on the left implies that $p_i - p_{i+1} \leq p_{i-1} - p_i$ which proves part (ii) of the proposition.

From Proposition 6 and equation (29), we know that $1 \leq \beta_i \leq 2$. On the other hand, equation (29) implies that

$$2 - \frac{1}{\beta_i} \leq \beta_{i-1} \leq 2 \quad (i = 2, \ldots, N - 1).$$

Using these two inequalities iteratively, we get a lower bound for $\beta_{i-1}$ with the following continued-fraction representation

$$2 \geq \beta_{i-1} \geq 2 - \frac{1}{\beta_i} \geq 2 - \frac{1}{2 - \frac{1}{\beta_{i+1}}} \geq 2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{\beta_{i+2}}}} \geq \cdots \geq 2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{\beta_{N-1}}}}}.$$ 

For $i = N$, equation (29) implies that $\beta_{N-1} = 2$ and so we can explicitly compute the continued-fraction above. After some straightforward manipulations, we get that

$$1 + \frac{1}{N + 1 - i} \leq \beta_{i-1} \leq 2 \quad (i = 2, \ldots, N - 1).$$

□

PROOF OF PROPOSITION 4. Rather than deriving the optimal solution $p(u)$ directly, we find
convenient to first compute its inverse \( u(p) \) by solving

\[
W(\infty) = \sup_{u(p), \bar{p}} \int_{\bar{p}}^{p} p F_p(p, u(p) - \alpha p) \, dp + \bar{p} \left[ F(\infty, \bar{u} - \alpha \bar{p}) - F(\bar{p}, \bar{u} - \alpha \bar{p}) \right]
\]

subject to \( \dot{u}(p) \geq \alpha \), \( u(p) = \bar{u} \) and \( u(\bar{p}) = \bar{u} \),

where \( \dot{u} \) is the first derivative of \( u(p) \) with respect to \( p \) and \( \bar{p} \) and \( \bar{p} \) are two additional unknowns. The constraint \( \dot{u}(p) \geq \alpha \) is the continuous version of the optimality condition \( (u_{i-1} - u_i) \geq \alpha (p_{i-1} - p_i) \) that derived in Proposition 6. Note that integrating this constraint between \( \bar{p} \) and \( \bar{p} \) leads to the additional requirement that \( \bar{u} - \bar{u} \geq \alpha (\bar{p} - \bar{p}) \).

We can solve the optimization problem above by inspection (alternatively, we could rely on Euler’s optimality condition, e.g., Fleming and Rishel (1975)). Indeed, note that for fixed value of \( \bar{p} \), the optimization reduces to

\[
W(\infty) = \sup_{u(p), \bar{p}} \int_{\bar{p}}^{p} p F_p(p, u(p) - \alpha p) \, dp, \quad \text{subject to} \quad \dot{u} \geq \alpha, \quad u(p) = \bar{u} \quad \text{and} \quad u(\bar{p}) = \bar{u}.
\]

Since the integrand \( p F_p(p, u - \alpha p) \) nonnegative and is increasing in \( u \), we would like to make \( \bar{p} \) as small as possible and \( u(p) \) as large as possible for every \( p \in [\bar{p}, \bar{p}] \). This observation together with the constraints \( \dot{u}(p) \geq \alpha \) and \( u(\bar{p}) = \bar{u} \) lead to an optimal solution

\[
p = \left( \bar{p} - \frac{\bar{u} - \bar{u}}{\alpha} \right)^+ \quad \text{and} \quad u(p) = \bar{u} - \alpha (\bar{p} - p), \quad p \in (\bar{p}, \bar{p}].
\]

Note that the pricing rule is not necessarily right continuous at \( \bar{p} \) if \( \bar{p} = 0 \). Also, it is straightforward to invert this solution to express the price as a function of the intrinsic utility

\[
p(u) = \left[ \bar{p} - \frac{\bar{u} - \bar{u}}{\alpha} \right]^+, \quad u \in [\bar{u}, \bar{u}].
\]

Recall that this solution assumes that \( \bar{p} \) is given. To find the optimal value of \( \bar{p} \), we replace the previous solution in \( W(\infty) \) to get the following one-dimensional optimization

\[
W(\infty) = \max_{\bar{p}} \int_{\bar{p}}^{p} p F_p(p, \bar{u} - \alpha \bar{p}) \, dp + \bar{p} \left[ F(\infty, \bar{u} - \alpha \bar{p}) - F(\bar{p}, \bar{u} - \alpha \bar{p}) \right] \quad \text{s.t.} \quad \bar{p} = \left( \bar{p} - \frac{\bar{u} - \bar{u}}{\alpha} \right)^+.
\]

Finally, we can use integration by parts to recover the optimization displayed in Proposition 4.

**Proof of Proposition 6** Consider an arbitrary pricing policy \( p_S(t) \) together with its corresponding inventory process \( I_S(t) \). Based on \( p_S(t) \) and \( I_S(t) \) we will define a new price process \( \hat{p}_S(t) \in \mathcal{P}_S \) with an associated inventory process \( \hat{I}_S(t) \) such that

\[
\int_{0}^{T} p_S(t) \cdot dI_S(t) = \int_{0}^{T} \hat{p}_S(t) \cdot d\hat{I}_S(t).
\]
This result ensures that we can restrict the search for an optimal pricing policy to the set \( \mathcal{P}_S \). Let us fix a time \( t \in [0,T] \) and define \( \hat{p}_0(t) = \hat{p}_0(t) \triangleq \max\{p_j(t) : j \in S\} \). We construct the price vector \( \hat{p}_S(t) \) using the following algorithm:

1. Set \( i = 1 \).
2. If \( I_i(t) = 0 \) then set \( \hat{p}_i(t) = \hat{p}_{i-1}(t) \). Otherwise, if \( I_i(t) > 0 \) then \( \hat{p}_i(t) = \min\{\hat{p}_{i-1}(t), p_i(t)\} \).
3. Set \( i = i + 1 \). If \( i < N + 1 \) goto 2, otherwise goto 4.
4. Set \( \hat{p}_N(t) = \hat{p}_N(t) \), set \( j = N - 1 \).
5. \( \hat{p}_j(t) = \min\{\hat{p}_j(t), \hat{p}_{j+1}(t) + \psi_{j+1}\} \), where \( \psi_{j+1} = \frac{u_j - u_{j+1}}{\alpha} \).
6. Set \( j = j - 1 \). If \( j > 0 \) go to 5, otherwise stop.

It follows from this algorithm that the resulting price process \( \hat{p}_S(t) \) belongs to \( \mathcal{P}_S \). In order to prove that condition \( 30 \) holds, we will show that for every realization of the demand process, \( I_S(t) = I_S(t) \) and \( p_S(t) \cdot dI_S(t) = \hat{p}_S(t) \cdot d\hat{I}_S(t) , \) for every \( t \).

Let us first suppose by contradiction that \( I_S(t) \neq \hat{I}_S(t) \). Then there exists a time \( \tau \in (0,T] \) such that \( \tau = \inf\{t > 0 : dI_S(t) \neq d\hat{I}_S(t)\} \). At this time \( \tau \) there is a customer arriving and buying some product \( i \). By construction \( \hat{p}_S(\tau) \leq p_S(\tau) \) and so we must have \( dI_S(\tau) = 0 \) and \( d\hat{I}_S(\tau) = -1 \). In other words, at time \( \tau \) the inventory levels under \( p_S(t) \) and \( \hat{p}_S(t) \) are the same and at time \( \tau \) they differ. This means that the customer who arrives at time \( \tau \) is willing to purchase a product \( i \) under \( \hat{p}_S(\tau) \) but he/she is not willing to purchase \( i \) under \( p_S(\tau) \). For this event to happen, we must have that \( I_i(\tau-) > 0 \) and \( \hat{p}_i(\tau) < p_i(\tau) \). So, according to the algorithm above, we must have one of the following two situations: (i) \( \hat{p}_i(\tau) = \hat{p}_{i-1}(\tau) \), (ii) \( \hat{p}_i(\tau) = \hat{p}_{i+1}(\tau) + \psi_{i+1} \). Notice that this second situation is not possible because it contradicts the assumption that the incoming customer buys product \( i \); if \( \hat{p}_i(\tau) = \hat{p}_{i+1}(\tau) + \psi_{i+1} \) (which assumes \( \hat{I}_{i+1}(\tau-) > 0 \), because if not, \( \hat{p}_{i+1}(\tau) \) would be equal to \( \hat{p}_i(\tau) \)) the customer would buy product \( i + 1 \) instead of product \( i \) since \( \hat{u}_{i+1} = \hat{u}_i \) and \( \hat{p}_{i+1}(\tau) < \hat{p}_i(\tau) \). In this way, we must have that \( \hat{p}_i(\tau) = \hat{p}_{i-1}(\tau) \). But if this is the case, it must be that \( \hat{I}_{i-1}(\tau-) = 0 \), otherwise the arriving buyer would prefer to get product \( i - 1 \) instead of product \( i \) since they are selling at the same price \( \hat{p}_i(\tau) = \hat{p}_{i-1}(\tau) \). But by the definition of \( \tau \), \( \hat{I}_{i-1}(\tau-) = I_{i-1}(\tau-) \) it follows that \( \hat{p}_{i-1}(\tau) = \hat{p}_{i-2}(\tau) \). Again, this condition implies that \( \hat{I}_{i-2}(\tau-) = 0 \), otherwise the arriving buyer would have purchased product \( i - 2 \) instead of \( i \).

Applying this reasoning recursively, we can show that if the arriving customer purchases product \( i \) then \( \hat{I}_j(\tau-) = I_j(\tau-) = 0 \) for all \( j = 1, \ldots, i - 1 \). But if this is the case, the algorithm implies that \( \hat{p}_i(\tau) = \hat{p}_0(\tau) \) for \( j = 1, \ldots, i \). But this contradicts our previous conclusion that \( \hat{p}_i(t) < p_i(t) \) since \( \hat{p}_i(\tau) \geq p_i(\tau) \). Hence, we conclude that \( I_S(t) = \hat{I}_S(t) \) for all \( t \).
Finally, we prove that \( p_S(t) \cdot dI_S(t) = \tilde{p}_S(t) \cdot d\tilde{I}_S(t) \). The condition holds trivially if \( d\tilde{I}_S(t) = dI_S(t) = 0 \). Now suppose that at time \( t \) there is a sale such that \( d\tilde{I}_i(t) = dI_i(t) = -1 \) and \( p_i(t) > \tilde{p}_i(t) \). This condition requires that \( I_i(t-) > 0 \) and according to the algorithm it also implies \( \tilde{p}_i(t) = \tilde{p}_{i-1}(t) \). Once again, we can use the same argument of the previous paragraph to conclude that we must have \( \tilde{p}_j(t) = \tilde{p}_0(t) \) for all \( j = 1, \ldots, i \), which contradicts \( p_i(t) > \tilde{p}_i(t) \). Hence, it follows that \( p_S(t) \cdot dI_S(t) = \tilde{p}_S(t) \cdot d\tilde{I}_S(t) \). □

Proof of Lemma 2

The dependence of \( W_k(p; Z, I_S) \) on \( Z \) follows directly from the definition of this auxiliary value function. Also, the fact that \( W_k(p; Z, I_S) \) is increasing in \( p \) follows from the fact that \( q_k(p, p_k) \) is increasing in \( p \). To prove the monotonicity properties of \( p^*_k(p, Z) \) note that the function

\[
G(p, p_k, z_k) = q_k(p, p_k) [p_k - z_k] + W_{k+1}(p_k, Z)
\]

satisfies

\[
\frac{\partial^2 G}{\partial p_k \partial z_k} \geq 0 \quad \text{and} \quad \frac{\partial^2 G}{\partial p_k \partial p} \geq 0.
\]

The second inequality uses Assumption 1. These inequalities, together with the fact that the set \( A_k(p) = [p - \psi_k, p] \) is increasing in \( p \), imply (by monotone comparative statics) that \( p^*_k(p, Z) \) increases in both \( p \) and \( z_k \). □

Proof of Theorem 1

First, to show that \( V(t, I_S) \) increases in \( I_i \) for all \( i = 1, \ldots, N \), we use a pathwise argument to show that \( V(t, I_S + e_i) \geq V(t, I_S) \). Indeed, the result follows by noticing that the optimal pricing policy that generates \( V(t, I_S) \) is also feasible when the initial state is \( (t, I_S + e_i) \) instead.

For any \( I_S \geq 0 \) and \( t \in [0, T] \) define the \( N \)-dimensional vector \( Z(t, I_S) = (Z_i(t, I_S)) \) as follows

\[
Z_i(t, I_S) \triangleq \mathbb{I}(I_i > 0) \left[ V(t, I_S) - V(t, I_S - e_i) \right], \quad i = 1, \ldots, N.
\]

Note that the HBJ equation (18) can be written as

\[
\frac{\partial V(t, I_S)}{\partial t} = \Phi(Z(t, I_S), I_S).
\]

By Lemma 1 \( \Phi \) is nonnegative and so \( V(t, I_S) \) is increasing in \( t \). Let us recall the definition of the \( N \)-dimensional vector \( \Delta V(t, I_S) = (\Delta_i V(t, I_S)) \), where \( \Delta_i V(t, I_S) = V(t, I_S) - V(t, I_S - e_i) \). We also define an auxiliary vector \( G(t, I_S) = (g_i(t, I_S)) \) such that \( g_i(t, I_S) = \Phi(\Delta V(t, I_S - e_i), I_S - e_i) \).

With these definitions, it is not hard to show (after differentiating with respect to \( t \)) that the vector
\( \Delta V(t, I_S) \) satisfies the \( N \)-dimensional system of first-order differential equation

\[
\frac{\partial}{\partial t} \Delta V(t, I_S) - \Phi(\Delta V(t, I_S), I_S) e = -G(t, I_S),
\]

with border condition \( \Delta V(0, I_S) = 0 \). In this previous expression \( e \) is an \( N \)-dimensional vector of ones. We now use the system of ODEs in (31) recursively and the monotonicity of \( \Phi(Z) \) to prove by induction that \( \Delta V(t, I_S) \) decreases in every component of \( I_S \) for all \( t \). The key step in this prove is the following auxiliary result.

**Lemma 3** Given two \( N \)-dimensional continuous functions \( H^1(t) \) and \( H^2(t) \), define two \( N \)-dimensional functions \( Z^1(t) \) and \( Z^2(t) \) as the solution of the following system of ODEs

\[
\frac{\partial}{\partial t} Z^j(t) - \Phi(Z^j(t)) e = H^j(t),
\]

with border condition \( Z^j(0) = 0 \). If \( H^1(t) \geq H^2(t) \) for all \( t \) then \( Z^1(t) \geq Z^2(t) \) for all \( t \).

Finally, from equation (31) we get that

\[
\frac{\partial}{\partial t} [V(t, I_S) - V(t, I_S - e_i)] = \Phi(\Delta V(t, I_S), I_S) - \Phi(\Delta V(t, I_S - e_i), I_S - e_i).
\]

This shows that \( V(t, I_S) - V(t, I_S - e_i) \) is increasing in \( t \) since \( \Phi(Z, I) \) is decreasing in \( Z \) and, as we have already shown, \( \Delta V(t, I_S) \leq \Delta V(t, I_S - e_i) \). □

**Proof of Proposition 7**

As in the proof of Proposition 2, we note that the prices \( p_i \) and Lagrangian multipliers \( \nu_i \) satisfy

- For \( \bar{p} = 0 \), \( p_i(\bar{p}) = 0 \) for all \( i = 1, \ldots, N \) and \( \nu_i(\bar{p}) = 0 \) for all \( i = 2, \ldots, N \).

- As \( \bar{p} \uparrow \infty \), \( p_i(\bar{p}) \uparrow \infty \) for all \( i = 1, \ldots, N \) and \( \nu_i(\bar{p}) \downarrow 0 \) for all \( i = 2, \ldots, N \).

Therefore, by the continuity of the distribution function \( F(p, u) \), we conclude that the range of the mapping \( p_i(\bar{p}) \ (i = i, \ldots, N) \) is the entire \( \mathbb{R}_+ \). This observation implies that the problem of existence reduces to find a price \( \bar{p} \in \mathbb{R}_+ \) such that the conditions in Step 3 of the algorithm are satisfied.

First of all, to ensure feasibility we need to select \( \bar{p} \) in such a way that

\[
\lambda T [F(p_0, u_1) - F(p_1(\bar{p}), u_1)] \leq I_1(0) \quad \iff \quad p_1(\bar{p}) \geq F^{-1} \left( F(p_0, u_1) - \frac{I_1(0)}{\lambda T}, u_1 \right).
\]

Since the right-hand side of the inequality is a constant, we can always select a \( \bar{p} \) that will satisfy this feasibility condition. Let us define \( \bar{p}^{\min} \) to be the price \( \bar{p} \) for which

\[
p_1^{\min} \triangleq p_1(\bar{p}^{\min}) = F^{-1} \left( F(p_0, u_1) - \frac{I_1(0)}{\lambda T}, u_1 \right).
\]
Recall that \( \hat{p} \) is the price that solves \( L(\hat{p}, u_1, u_2) = F(p_0, u_1) \). Suppose that \( \hat{p} \leq p_1^{\text{min}} \) then because of the unimodal assumption on \( L(p, u_1, u_2) \), it is not hard to see that

\[
F(p_0, u_1) - L(p_1^{\text{min}}, u_1, u_2) = F(p_0, u_1) - F(p_1^{\text{min}}, u_1) - p_1^{\text{min}} F_p(p_1^{\text{min}}, u_1) + p_1^{\text{min}} F_p(p_1^{\text{min}}, u_2) \leq 0.
\]

In addition, since \( p_1^{\text{min}} \geq p_2(\bar{p}^{\text{min}}) \) and \( \nu_2(\bar{p}^{\text{min}}) \geq 0 \) we conclude that

\[
F(p_0, u_1) - F(p_1^{\text{min}}, u_1) - p_1^{\text{min}} F_p(p_1^{\text{min}}, u_1) + (p_2(\bar{p}^{\text{min}}) - \nu_2(\bar{p}^{\text{min}})) F_p(p_1^{\text{min}}, u_2) \leq 0.
\]

Therefore, we can find an \( \nu_1 \geq 0 \) such that

\[
F(p_0, u_1) - F(p_1^{\text{min}}, u_1) - (p_1^{\text{min}} - \nu_1) F_p(p_1^{\text{min}}, u_1) + (p_2(\bar{p}^{\text{min}}) - \nu_2(\bar{p}^{\text{min}})) F_p(p_1^{\text{min}}, u_2) = 0
\]

and so by choosing \( \bar{p} = \bar{p}^{\text{min}} \) the algorithm generates a solution to the KKT condition.

Suppose, on the other hand, that \( \hat{p} \geq p_1^{\text{min}} \) then we can repeat the previous argument but replacing \( p_1^{\text{min}} \) by \( \hat{p} \), and we can still find a \( \nu_1 \geq 0 \), such that the following equality holds

\[
F(p_0, u_1) - F(\hat{p}, u_1) - (\hat{p} - \nu_1) F_p(\hat{p}, u_1) + (p_2(\bar{p}^{\text{min}}) - \nu_2(\bar{p}^{\text{min}})) F_p(\hat{p}, u_2) = 0.
\]

This completes the proof of the sufficient condition.

To prove the necessity part, we simply notice that

\[
F(p_0, u_1) - U(p_1, u_1) \leq F(p_0, u_1) - F(p_1, u_1) - (p_1 - \nu_1) F_p(p_1, u_1) + (p_2 - \nu_2) F_p(p_1, u_2)
\]

for all \( p_1 \geq p_2 \) and \( \nu_1 \geq 0 \) and \( p_2 \geq \nu_2 \geq 0 \). The fact that \( p_2 \geq \nu_2 \) follows from Step 2 in the algorithm. Therefore, if there is a solution to the KKT conditions then the right-hand side of the inequality would be equal to zero for this solution. This implies that the left-hand side would less than or equal to zero for this solution. But for \( p_1 = 0 \) the left hand side is strictly positive. Thus by continuity we conclude that there has to be a \( \tilde{p} \) such that \( F(p_0, u_1) = U(\tilde{p}, u_1) \). □