Appendix

Proof of Theorem 1:

From Lemma 1, the time available for attending to the long term process is given by

$$1 - \sum_{i=1}^{n} \frac{d_i}{u_i + d_i},$$

where $n$ is the number of short term processes. Assume that during the remaining time the manager devotes attention to the long term process. We shall establish the necessity of this condition first by contradiction. Assume that there is a constant, $\varepsilon > 0$, such that

$$1 - \sum_{i=1}^{n} \frac{d_i}{u_i + d_i} = \frac{d_0 + u_0}{(d_0 + u_0)} - \varepsilon. \tag{A1}$$

Therefore, from Lemma 1 and the strong law of large numbers,

$$N_U^0(t) = \left(\frac{d_0 + u_0}{(d_0 + u_0)} - \varepsilon\right) t + o(t). \tag{A2}$$

From equations (2) and (A2),

$$\lim_{t \to \infty} \frac{N_U^0(t)}{(N_U^0(t) + N_D^0(t))} = \lim_{t \to \infty} \frac{N_U^0(t)}{t} = \left(\frac{d_0 + u_0}{(d_0 + u_0)} - \varepsilon\right). \tag{A3}$$

Using the logic depicted in Fig. 1 and equations (2) and (A3), the rate of increase of the level of process zero is given by

$$\lim_{t \to \infty} \frac{N_U^0(t)}{t} = \lim_{t \to \infty} \frac{N_U^0(t)}{t} - d_0 = \lim_{t \to \infty} \frac{N_U^0(t)}{t} - d_0 + u_0) - d_0 = \varepsilon - a_0. \tag{9}$$

This is a contradiction. Therefore the condition given in equation (9) is necessary. We prove its sufficiency by producing a rule that achieves stability as well as the long term goal. Assume that the manager assigns the highest priority to the short term process 1, the second highest priority to process 2, so forth, and thereby assigns the lowest priority to process $n$. A short term process is said to require the manager’s attention if its level falls below 1. The manager follows the rule of immediately attending to the short term process with the highest priority that needs attention. Once all short term processes reach level 1, the manager devotes attention to the long term process. We shall prove that this rule provides stability and is also one that achieves the long term goal.

We need a few intermediate results regarding the periods during which the manager attends only to the first $k$ processes (called the $k$-busy period). Let all short term processes be at level 1 at
time zero. Denote the $j$th interval of time during which the manager devotes undivided attention to the set of processes $\{1, 2, \ldots, k\}$ as the $j$th k-busy period, $T^{(k)}_j$. Denote the interval between the end of the $j$th k-busy period and the commencement of the $(j+1)$st k-busy period as the $(j+1)$st k-idle period, $I^{(k)}_{j+1}$. Define,

$$p^{(k)} = \prod_{i=1}^{k} (1 - d_i), \quad q^{(k)} = 1 - p^{(k)}.$$  \hspace{1cm} (A4)

From the rule described above it follows that the $k$-idle periods are i.i.d. random variables that are geometrically distributed with

$$\Pr\{I^{(k)}_j = m\} = (p^{(k)})^{k-1} q^{(k)}.$$ \hspace{1cm} (A5)

Therefore, the $k$-idle period has finite expectation that is equal to,

$$E[I^{(k)}_1] = \frac{1}{q^{(k)}}.$$ \hspace{1cm} (A6)

The proof consists of showing that for any $k$, the $k$-busy periods are also i.i.d. random variables and that they have finite expectation. The proof of sufficiency will follow from this fact. It is easy to verify that the $k$-busy periods are i.i.d. random variables. Notice that while this claim might appear to be counterintuitive, the states of the first $k$ short term processes are restored to the same value at the end of the $j$th k-busy period. Therefore, probabilistically the evolution of these $k$ processes beyond the end of the $j$th k-busy period is identical to their evolution beyond the end of the $(j-1)$st busy period. Define, $\{Y_l, l = 1, 2, \ldots\}$ to be i.i.d. random variables that take the value 1 with probability $d_{k+1}$ and the value zero with probability $(1-d_{k+1})$. Let $\{X_l, l = 1, 2, \ldots\}$ be i.i.d. random variables that take the value 1 with probability $u_{k+1}$ and the value zero with probability $(1-u_{k+1})$. Define

$$Z^{(k)}_j = \sum_{l=1}^{I^{(k)}_j} Y_l - \sum_{l=1}^{I^{(k)}_{j+1}} X_l.$$ \hspace{1cm} (A7)

The random variables $\{Y_l, l = 1, 2, \ldots\}$ and $\{X_l, l = 1, 2, \ldots\}$ are mutually independent sequences of i.i.d. random variables that are also independent of the $k$-busy and $k$-idle periods.
Therefore, the $Z_j^{(k)}$'s are i.i.d. random variables. We are now ready for the final result. We show by induction on $k$ that the following two claims hold

$$E[T_1^{(k)}] < \infty, \quad \text{and}$$

$$\exists \varepsilon > 0 : E[e^{\theta Z_1^{(k)}}] < \infty \quad \text{for all } 0 \leq \theta \leq \varepsilon,$$

(A8)

Both these claims are true when $k$ equals 1, because the manager works on the first process whenever its level drops to zero, and attends to it until its level reaches 1. Assume that the two claims are true for $k$. From the induction hypothesis and the use of Wald's Lemma (see Ross [1983]), $E[Z_1^{(k)}]$ is finite. Moreover, from the proof that condition (9) is necessary, it also follows that the fraction of time spent by the manager attending to the first $k$ short term processes, denoted by $f_k$, is given by

$$f_k = \sum_{i=1}^k \frac{d_i}{d_i + u_i}. \tag{A10}$$

By using the renewal reward theorem (see Ross [1983]), we obtain

$$f_k = \frac{E[T_1^{(k)}]}{E[T_1^{(k)}] + E[T_1^{(1)}]} \tag{A11}.$$

Equations (A6), (A10), and (A11) imply that

$$E[T_1^{(k)}] = \left( \sum_{i=1}^k \frac{d_i}{d_i + u_i} \right) / q^{(k)} \left( 1 - \left( \sum_{i=1}^k \frac{d_i}{d_i + u_i} \right) \right). \tag{A12}$$

From equations (A6), (A7), (A12), and Wald's Lemma

$$E[Z_1^{(k)}] = \frac{\left( \sum_{i=1}^{k+1} \frac{d_i}{d_i + u_i} \right)}{q^{(k+1)} \left( 1 - \left( \sum_{i=1}^k \frac{d_i}{d_i + u_i} \right) \right)}. \tag{A13}$$

From equation (A13), $E[Z_1^{(k)}]$ is strictly less than zero. From equation (A7) we obtain

$$\Pr\{T_1^{(k+1)} > \sum_{j=1}^m (T_j^{(k)} + I_j^{(k)}) \} \leq \Pr\{ \sum_{j=1}^m Z_j^{(k)} \geq 1 \},$$

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because the event \( \{ T_1^{(k+1)} > \sum_{j=1}^{m} (T_j^{(k)} + I_j^{(k)}) \} \) implies the event \( \{ \sum_{j=1}^{m} Z_j^{(k)} \geq 1 \} \). From Theorem 1.5 of Shwartz and Weiss [1995], and because \( E[Z_1^{(k)}] < 0 \), there exists a suitable constants \( K > 0 \) and \( C > 0 \), such that

\[
Pr\{ \sum_{j=1}^{m} Z_j^{(k)} \geq 1 \} \leq Ke^{-mc}. \tag{A14}
\]

It follows from equation (A14) that both the claims, namely equations (A8) and (A9) of the induction hypothesis are true for \( (k+1) \). Finally, from equations (A6) and (A12), we obtain that the ratio,

\[
\lim_{m \to \infty} \frac{\sum_{j=1}^{m} d_j^{(n)}}{\sum_{j=1}^{m} I_j^{(n)}} = 1 - \sum_{i=1}^{n} \frac{d_i}{d_i + u_i}. \tag{A15}
\]

Equations (9) and (A15) imply that the manager is able to devote sufficient time to the long term process in order to achieve the long term goal.

The levels of the short term processes are equal to one when working on the long term process. When interrupted from working on the long term process the worst case happens when all their levels simultaneously fall to zero. This observation and (A14) imply that condition (3) holds.

It may be noted that the initial conditions assumed in the proof of sufficiency are not that important. We only require that the expected duration of the first \( n \)-busy period is finite. \[\blacksquare\]

**Proof of Theorem 2**: Consider a duration of time \( T \). During this time, let \( T_{av} \) be the time available to work on the long term process. Based on the proof of Theorem 1

\[
T_{av} = T(1 - \sum_{i=1}^{n} \frac{d_i}{u_i + d_i}) + o(T). \tag{A16}
\]

The average available time does not depend on the long term process parameters, \( u_0 \) and \( d_0 \), as long as we assume that the short term processes have priority over the long term process and that the short term processes are stable. Assume that the time that is not spent attending to the short term processes is entirely expended in working on the long term process. Denote the average
duration of a busy period working on the long term process as $T^{(0)}$. Then the average number of interruptions during time $T$ is given by $T_{av}/T^{(0)}$. In particular, this number is independent of the distribution of the periods during which the manager is not working on the long term process. We make three observations:

(i) During the interval of time $[0, T]$, the manager attends to the long term process for a duration $T_{av}$. Therefore, due to this effort alone, the expected level of the long term process increases by the amount, $u_0 T_{av}$.

(ii) Similarly, during $T - T_{av}$, the long term process deteriorates by the average amount, $d_0(T - T_{av})$.

(iii) The expected downward drift due to interruptions is given by the expected number of interruptions times the cost of interruptions, namely, $(T_{av}/T^{(0)}) \Delta$.

Combining these three observations and using equation (A16), we conclude that in order to achieve the target growth rate of $a_0$, we must have

\[ a_0 T \leq u_0 T \left( 1 - \sum_{i=1}^{n} \frac{d_i}{u_i + d_i} \right) - d_0 T \left( \sum_{i=1}^{n} \frac{d_i}{u_i + d_i} \right) - \Delta T \left( 1 - \sum_{i=1}^{n} \frac{d_i}{u_i + d_i} \right)/T^{(0)} \]

\[ \Rightarrow a_0 + d_0 + \Delta \left( 1 - \sum_{i=1}^{n} \frac{d_i}{u_i + d_i} \right) \leq u_0 \left( 1 - \sum_{i=1}^{n} \frac{d_i}{u_i + d_i} \right) + d_0 \left( 1 - \sum_{i=1}^{n} \frac{d_i}{u_i + d_i} \right) \]

\[ \Rightarrow 1 - \sum_{i=1}^{n} \frac{d_i}{u_i + d_i} \geq \frac{a_0 + d_0 + \Delta \left( 1 - \sum_{i=1}^{n} \frac{d_i}{u_i + d_i} \right)}{u_0 + d_0}. \]  
(A17)

Rearranging the terms in (A17) we obtain, $T^{(0)} \geq \frac{\Delta}{u_0 + d_0 - \frac{\sum_{i=1}^{n} d_i}{u_i + d_i}}$.  

\[ \blacksquare \]