316-466 Monetary Economics — Homework 1

Consider the following problem. Time is discrete and denoted $t = 0, 1, ..., T$ for some finite horizon T. There is no uncertainty. A single consumer has preferences over consumption c and leisure ℓ given by

$$U(c, \ell) = \sum_{t=0}^{T} \beta^t u(c_t, \ell_t) \qquad 0 < \beta < 1$$

The utility function u is assumed to be strictly increasing and strictly concave in both arguments.

She owns some initial physical capital $k_0 > 0$. Each period t, she either consumes c_t or invests i_t subject to a budget constraint

$$c_t + i_t \leq y_t \equiv w_t n_t + r_t k_t \tag{1}$$

Her income y_t is comprised of labor income (real wage times labor supplied) $w_t n_t$, plus capital income (rental rate times capital supplied) $r_t k_t$. She supplies labor to a competitive spot market subject to the constraint that the total time consuming leisure plus working satisfy

$$\ell_t + n_t \leq 1$$

each period. Thus n_t is interpreted as her fraction of time working.

She supplies capital to a competitive spot market. Any investment she does augments her capital stock according to

$$k_{t+1} = (1 - \delta)k_t + i_t, \qquad 0 < \delta < 1 \tag{2}$$

where δ denotes the depreciation rate of physical capital.

Now:

1. Formulate a Lagrangian involving the sequence of budget constraints (1) and the sequence of capital accumulation equations (2) and use this Lagrangian to characterize her optimal capital accumulation $\{k_{t+1}\}_{t=0}^{T}$ and labor supply decisions $\{n_t\}_{t=0}^{T}$. Why is $k_{T+1} = 0$? Give economic interpretations to the consumer's first order conditions. Is the real interest rate equal to the rental rate of capital, r_t? If not, why not.

2. Combine (1) and (2) and construct her intertemporal budget constraint by recursive substitution. Write this intertemporal constraint using the price p_t of the dated consumption good c_t. This requires you to specify what p_t is. Formulate a new Lagrangian with the intertemporal budget constraint and compare the first order conditions you derive with those obtained in Question 1.

3. Now consider a representative firm that hires capital and labor from the consumer. This firm operates a production function F

$$y_t \leq F(k_t, n_t)$$

which is strictly increasing, strictly concave and exhibits constant returns to scale in capital and labor. The firm seeks to maximize the discounted value of its profits

$$\sum_{t=0}^{T} p_t[F(k_t, n_t) - w_t n_t - r_t k_t]$$

Derive and give economic interpretation to the first order conditions for the firm's choice of $\{k_t\}_{t=0}^{T}$ and $\{n_t\}_{t=0}^{T}$. Show that the firm's problem is equivalent to maximizing static profits

$$F(k_t, n_t) - w_t n_t - r_t k_t$$

by choice of k_t and n_t each period. Show that the constant returns to scale property of the production function F implies that there are indeed *zero* economic profits each period.

4. Modify the model so that the consumer does not own capital and does not accumulate capital. She just works and consumes each period. Now, however, the firm owns the initial capital stock $k_0 > 0$ and does its own capital accumulation (according to (2)). What are the firm's static profits each period? What is the firm's intertemporal optimization problem? Derive first order conditions characterizing the firm's optimal choice of $\{k_{t+1}\}_{t=0}^{T}$ and $\{n_t\}_{t=0}^{T}$ for this new problem, taking as given the intertemporal prices p_t.

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