316-466 Monetary Economics — Midsemester Solutions

1. **Welfare costs of inflation**. (30 marks).

   (a) (5 marks). The Lagrangian for this problem can be written

$$L = \sum_{t=0}^{\infty} \beta^t U\left(c_t, \frac{M_{t+1}}{P_t}\right) + \sum_{t=0}^{\infty} \lambda_t (P_t y_t + M_t + B_t - T_t - P_t c_t - M_{t+1} - q_t B_{t+1})$$

   The key FONC include, for choice of consumption, money, and bonds,

$$\begin{aligned} \beta^t \frac{U_c(c_t, m_t)}{P_t} &= \lambda_t \\ \beta^t \frac{U_m(c_t, m_t)}{P_t} + \lambda_{t+1} &= \lambda_t \\ \lambda_{t+1} &= \lambda_t q_t \end{aligned}$$

   Combining these gives

$$\begin{aligned} q_t &= \beta \frac{U_c(c_{t+1}, m_{t+1})}{U_c(c_t, m_t)} \frac{P_t}{P_{t+1}} \\ 1 - q_t &= \frac{U_m(c_t, m_t)}{U_c(c_t, m_t)} \end{aligned}$$

   Using the definition of the net nominal interest rate, the latter implies

$$\frac{U_m(c_t, m_t)}{U_c(c_t, m_t)} = \frac{i_t}{1 + i_t}$$

   so that the MRS between real balances and consumption is set equal to the opportunity cost $i_t/(1+i_t)$ of holding money (instead of nominal bonds).

   (b) (5 marks). Given the definition of the utility function,

$$U(c, m) = \frac{1}{1-\sigma}\left\{\left[c\varphi\left(\frac{m}{c}\right)\right]^{1-\sigma} - 1\right\}, \qquad \sigma > 0$$

   the marginal utilities are

$$\begin{aligned} U_m(c, m) &= \left[c\varphi\left(\frac{m}{c}\right)\right]^{-\sigma} \varphi'\left(\frac{m}{c}\right) \\ U_c(c, m) &= \left[c\varphi\left(\frac{m}{c}\right)\right]^{-\sigma} \left[\varphi\left(\frac{m}{c}\right) - \varphi'\left(\frac{m}{c}\right)\frac{m}{c}\right] \end{aligned}$$

   Taking ratios of these two marginal utilities and simplifying

$$\begin{aligned} \frac{i_t}{1+i_t} &= \frac{U_m(c_t, m_t)}{U_c(c_t, m_t)} = \frac{\left[c_t\varphi\left(\frac{m_t}{c_t}\right)\right]^{-\sigma} \varphi'\left(\frac{m_t}{c_t}\right)}{\left[c_t\varphi\left(\frac{m_t}{c_t}\right)\right]^{-\sigma} \left[\varphi\left(\frac{m_t}{c_t}\right) - \varphi'\left(\frac{m_t}{c_t}\right)\frac{m_t}{c_t}\right]} \\ &= \frac{\varphi'\left(\frac{m_t}{c_t}\right)}{\varphi\left(\frac{m_t}{c_t}\right) - \varphi'\left(\frac{m_t}{c_t}\right)\frac{m_t}{c_t}} = \frac{\varphi'(x_t)}{\varphi(x_t) - \varphi'(x_t)\, x_t} \end{aligned}$$

(c) (5 marks). Steady state welfare is maximized when $U(1,m)$ is maximized, i.e., when $U_m(1,m) = 0$ (since $U(1,m)$ is strictly concave in m, the maximum problem is completely characterized by the necessary first order condition). But given the demand for money condition,

$$\frac{U_m(1,m)}{U_c(1,m)} = \frac{\varphi'(m)}{\varphi(m) - \varphi'(m)\,m} = \frac{i}{1+i}$$

and that the marginal utility of consumption is strictly positive for all m, welfare is maximized when $i = 0$. The $m^\star$ solves

$$\varphi'(m^\star) = 0$$

(d) (5 marks). Let

$$\varphi(x) = \frac{1}{1 + (B/x)}, \qquad B > 0$$

Then

$$\varphi'(x) = \left(\frac{1}{1+(B/x)}\right)^2 \frac{B}{x^2}$$

Using the first order condition for money demand evaluated at $(c,m) = (1,m)$,

$$\frac{i}{1+i} = \frac{\varphi'(m)}{\varphi(m) - \varphi'(m)\,m} = \frac{B}{m^2}$$

Hence

$$m^2 = B\left(\frac{i}{1+i}\right)^{-1}$$

or

$$m = B^{1/2}\left(\frac{i}{1+i}\right)^{-1/2} \equiv \hat{m}(i)$$

Clearly, $m^\star = \hat{m}(0) = +\infty$. Since $\varphi(m^\star) = \varphi(\infty) = 1$, we have

$$\begin{aligned} U(1,m^\star) &= U(1,\infty) \\ &= \frac{1}{1-\sigma}\left\{[\varphi(\infty)]^{1-\sigma} - 1\right\} = \frac{1}{1-\sigma}\left\{1^{1-\sigma} - 1\right\} = 0 \end{aligned}$$

(e) (5 marks). Let $\hat{w}(i)$ solve

$$U[1 + \hat{w}(i), \hat{m}(i)] = U(1,m^\star) = 0$$

where

$$\hat{m}(i) = B^{1/2}\left(\frac{i}{1+i}\right)^{-1/2}$$

Given the assumed utility function,

$$
\begin{aligned}
U[1+\hat{w}(i),\hat{m}(i)] &= \frac{1}{1-\sigma}\left\{\left[(1+\hat{w}(i))\varphi\left(\frac{1+\hat{w}(i)}{\hat{m}(i)}\right)\right]^{1-\sigma}-1\right\}=0 \\
&\Longleftrightarrow (1+\hat{w}(i))\varphi\left(\frac{1+\hat{w}(i)}{\hat{m}(i)}\right)=1
\end{aligned}
$$

Now using the assumed functional form for $\varphi(x)$, this is equivalent to requiring

$$
\begin{aligned}
(1+\hat{w}(i))\frac{1}{1+B\frac{1+\hat{w}(i)}{\hat{m}(i)}} &= 1 \\
\Longleftrightarrow \hat{w}(i) &= \frac{B}{\hat{m}(i)-B}
\end{aligned}
$$

(f) (5 marks). Let $B=0.0018$. Then for $i=0.1$,

$$
\begin{aligned}
\hat{m}(0.1) &= B^{1/2}\left(\frac{0.1}{1.1}\right)^{-1/2}=(0.0018)^{1/2}\left(\frac{0.1}{1.1}\right)^{-1/2}=0.1406 \\
\hat{w}(0.1) &= \frac{B}{\hat{m}(0.1)-B}=\frac{0.0018}{0.1406-0.0018}=0.01296
\end{aligned}
$$

This is a welfare cost of 1.3% of steady state consumption.

2. **Asset pricing — Hansen-Jagannathan bounds**. (30 marks).

(a) (2 marks). The asset returns satisfy

$$
\begin{aligned}
1 &= E_t\left\{\beta\frac{U'(c_{t+1})}{U'(c_t)}R^e_{t+1}\right\} \\
1 &= E_t\left\{\beta\frac{U'(c_{t+1})}{U'(c_t)}R^f_{t+1}\right\}
\end{aligned}
$$

Subtracting one from the other,

$$
0=E_t\left\{\beta\frac{U'(c_{t+1})}{U'(c_t)}[R^e_{t+1}-R^f_{t+1}]\right\}
$$

Taking unconditional expectations on both sides,

$$
\begin{aligned}
0 &= E\left\{E_t\left[\beta\frac{U'(c_{t+1})}{U'(c_t)}(R^e_{t+1}-R^f_{t+1})\right]\right\} \\
&= E\left\{\beta\frac{U'(c_{t+1})}{U'(c_t)}[R^e_{t+1}-R^f_{t+1}]\right\}
\end{aligned}
$$

(b) (8 marks). Let $R_{t+1}=R^e_{t+1}-R^f_{t+1}$ and $m_{t+1}=\beta U'(c_{t+1})/U'(c_t)$. Then

$$
\begin{aligned}
0 &= E\{m_{t+1}R_{t+1}\} \\
&= E\{m_{t+1}\}E\{R_{t+1}\}+\text{Cov}\{m_{t+1},R_{t+1}\} \\
&= E\{m_{t+1}\}E\{R_{t+1}\}+\text{Corr}\{m_{t+1},R_{t+1}\}\text{Std}\{m_{t+1}\}\text{Std}\{R_{t+1}\}
\end{aligned}
$$

because $\text{Cov}(X,Y) \equiv E(XY)-E(X)E(Y)$ and $\text{corr}(X,Y) \equiv \text{Cov}(X,Y)/(\text{Std}(X)\text{Std}(Y))$. Rearranging,

$$\frac{E\{R_{t+1}\}}{\text{Std}\{R_{t+1}\}} = -\text{Corr}\{m_{t+1}, R_{t+1}\}\frac{\text{Std}\{m_{t+1}\}}{E\{m_{t+1}\}}$$

(c) (5 marks). Since the correlation coefficient must satisfy $-1 \leq \text{corr}\{m_{t+1}, R_{t+1}\} \leq 1$ and the Sharpe ratio of the equity premium is

$$\frac{E\{R_{t+1}\}}{\text{Std}\{R_{t+1}\}} = \frac{1}{2}$$

we must have

$$\begin{aligned} -\frac{1}{2}\frac{E\{m_{t+1}\}}{\text{Std}\{m_{t+1}\}} &= \text{Corr}\{m_{t+1}, R_{t+1}\} \leq 1 \\ -\frac{1}{2}\frac{E\{m_{t+1}\}}{\text{Std}\{m_{t+1}\}} &= \text{Corr}\{m_{t+1}, R_{t+1}\} \geq -1 \end{aligned}$$

Hence

$$\begin{aligned} \frac{\text{Std}\{m_{t+1}\}}{E\{m_{t+1}\}} &\geq -\frac{1}{2} \\ \frac{\text{Std}\{m_{t+1}\}}{E\{m_{t+1}\}} &\geq \frac{1}{2} \end{aligned}$$

Hence $\underline{m} = 0.5$ and $\overline{m} = +\infty$. The standard deviation of the SDF has to be at least half as large as its mean.

(d) (3 marks). The conditional mean of the stochastic discount factor must satisfy

$$1 = E_t\left\{m_{t+1}R^f_{t+1}\right\} = E_t\{m_{t+1}\}R^f_{t+1}$$

since R^f_{t+1} is risk free. Hence

$$E_t\{m_{t+1}\} = \frac{1}{R^f_{t+1}}$$

and

$$E\{E_t\{m_{t+1}\}\} = E\{m_{t+1}\} = E\left\{\frac{1}{R^f_{t+1}}\right\}$$

If the average safe return is about $E\{R^f_{t+1}\} = 1.01$, then the mean of the risk free rate must be (to a first order approximation) $1/1.01 = 0.99$.

(e) (5 marks). With CRRA preferences,

$$m_{t+1} = \beta\frac{U'(c_{t+1})}{U'(c_t)} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = \beta\exp\left(-\gamma\Delta\log c_{t+1}\right)$$

so

$$\text{Std}\{m_{t+1}\} = \beta\sigma\{\exp\left(-\gamma\Delta\log c_{t+1}\right)\}$$

(f) (7 marks). This answer goes some way beyond what I expected, which was basically "γ has to be huge". A (very crude) first order approximation comes in handy. Suppose we write

$$\exp(-\gamma x_{t+1}) \simeq \exp(-\gamma\bar{x}) - \gamma\exp(-\gamma\bar{x})(x_{t+1}-\bar{x})$$

where $x_{t+1} \equiv \Delta\log c_{t+1}$ is consumption growth and $\bar{x} = E\{\Delta\log c_{t+1}\}$ is its unconditional mean. Then taking variances on both sides

$$\begin{aligned}\mathrm{Var}\{\exp(-\gamma x_{t+1})\} &\simeq \mathrm{Var}\{\exp(-\gamma\bar{x})\} + \mathrm{Var}\{\gamma\exp(-\gamma\bar{x})(x_{t+1}-\bar{x})\} \\ &= \gamma^2\exp(-2\gamma\bar{x})\,\mathrm{Var}\{x_{t+1}\}\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Std}\{\exp(-\gamma x_{t+1})\} &= \sqrt{\mathrm{Var}\{\exp(-\gamma x_{t+1})\}} \\ &\simeq \gamma\exp(-\gamma\bar{x})\sqrt{\mathrm{Var}\{x_{t+1}\}} = \gamma\exp(-\gamma\bar{x})\,\mathrm{Std}\{x_{t+1}\}\end{aligned}$$

or

$$\mathrm{Std}\{m_{t+1}\} = \beta\mathrm{Std}\{\exp(-\gamma\Delta\log c_{t+1})\} \simeq \beta\gamma\exp(-\gamma E\{\Delta\log c_{t+1}\})\,\mathrm{Std}\{\Delta\log c_{t+1}\}$$

If the standard deviation of consumption growth is about $\mathrm{Std}\{\Delta\log c_{t+1}\} = 0.01$, the mean of consumption growth is $E\{\Delta\log c_{t+1}\} = 0.02$ and $\beta = 1$, then the standard deviation of the SDF as a function of γ is approximately

$$\mathrm{Std}\{m_{t+1}\}(\gamma) \simeq \gamma\exp(-\gamma(0.02))0.01$$

This is a strictly concave function in γ that reaches a maximum at around $\gamma = 50$ where

$$\mathrm{Std}\{m_{t+1}\}(50) \simeq 0.1839$$

Since the standard deviation of the SDF has to be at least half as large as its mean (which is about 0.99, see part d) above) this theory does a bad job of reconciling consumption and asset returns data. (I have under-sold it a little with the linear approximation above, but not much — even with second order terms thrown in, it takes a huge value of risk aversion to bring $\mathrm{Std}\{m_{t+1}\}$ up to values that lie inside the bound). This is a huge problem for consumption-based asset pricing models. In short, consumption is way too smooth to explain the volatility of the SDF that would explain a sizeable equity risk premium.

3. **Monetary economics with linear production** (30 marks).

(a) (2 marks). The FONC that characterizes interior solutions to the firm's problem can be written

$$w_t(s^t) = \frac{W_t(s^t)}{P_t(s^t)} = A_t(s^t)$$

The real wage $w_t(s^t)$ is equal to labor productivity.

(b) (5 marks). The Lagrangian can be written

$$
\begin{aligned}
L &= \sum_{t=0}^{\infty}\sum_{s^t}\beta^t U\left[c_t(s^t), \frac{M_{t+1}(s^t)}{P_t(s^t)}, 1-n_t(s^t)\right] f(s^t \mid s_0) \\
&\quad + \sum_{t=0}^{\infty}\sum_{s^t}\lambda_t(s^t)\Big[W_t(s^t)n_t(s^t) + M_t(s^{t-1}) + B_t(s^{t-1}, s_t) - T_t(s^t) \\
&\quad - P_t(s^t)c_t(s^t) - M_{t+1}(s^t) - \sum_{s'} q_t(s^t, s')B_{t+1}(s^t, s')\Big]
\end{aligned}
$$

where $\lambda_t(s^t) \geq 0$ denotes the Lagrange multiplier for date t given history s^t. The key FONC are, for consumption, money, labor supply, and state contingent bonds,

$$
\begin{aligned}
\lambda_t(s^t) &= \beta^t \frac{U_{c,t}(s^t)}{P_t(s^t)} f(s^t \mid s_0) \\
\lambda_t(s^t) &= \beta^t \frac{U_{m,t}(s^t)}{P_t(s^t)} f(s^t \mid s_0) + \sum_{s'} \lambda_{t+1}(s^t, s') \\
\lambda_t(s^t) &= \beta^t \frac{U_{\ell,t}(s^t)}{W_t(s^t)} f(s^t \mid s_0) \\
q_t(s^t, s')\lambda_t(s^t) &= \lambda_{t+1}(s^t, s')
\end{aligned}
$$

where I use the short-hand notation $U_{c,t}(s^t) \equiv U_c[c_t(s^t), m_t(s^t), \ell_t(s^t)]$ (and so on). These first order conditions can be re-written

$$
\begin{aligned}
\frac{U_{m,t}(s^t)}{U_{c,t}(s^t)} &= \frac{i_t(s^t)}{1+i_t(s^t)} \\
\frac{U_{\ell,t}(s^t)}{U_{c,t}(s^t)} &= \frac{W_t(s^t)}{P_t(s^t)} \\
q_t(s^t, s') &= \beta \frac{U_{c,t+1}(s^t, s')}{U_{c,t}(s^t)} \frac{P_t(s^t)}{P_{t+1}(s^t, s')} f(s' \mid s_t) \\
\frac{1}{1+i_t(s^t)} &= \sum_{s'} q_t(s^t, s')
\end{aligned}
$$

These are, respectively, conditions governing money demand, labor supply, the pricing of one-period state contingent bonds and the determination of the safe nominal interest rate.

(c) (5 marks). As above,

$$
\frac{1}{1+i_t(s^t)} = \sum_{s'} q_t(s^t, s') = \sum_{s'} \beta \frac{U_{c,t+1}(s^t, s')}{U_{c,t}(s^t)} \frac{P_t(s^t)}{P_{t+1}(s^t, s')} f(s' \mid s_t)
$$

is the price of a safe nominal bond (that pays one unit of account — say \$1 — irrespective of the state s' that realizes at date $t+1$). This bond is not safe in real terms, there is inflation risk.

(d) (8 marks). The non-stochastic steady state has money and prices growing at constant rate $\overline{\mu}$ such that real balances are constant

$$\begin{aligned} P_t &= (1+\overline{\mu})^t P_0 \\ M_t &= (1+\overline{\mu})^t M_0 \end{aligned}$$

where M_0 is a known initial condition but where P_0 is an endogenous variable that has to be determined. The real interest rate satisfies

$$1+\bar{r} = \frac{1}{\beta} = \frac{1+\bar{\imath}}{1+\overline{\mu}}$$

Hence the real interest rate is equal to the rate of time preference and the nominal interest rate is the real rate plus inflation

$$\begin{aligned} 1+\bar{r} &= \frac{1}{\beta} \\ 1+\bar{\imath} &= \frac{1+\overline{\mu}}{\beta} \end{aligned}$$

The real wage rate is simply

$$\bar{w} = \bar{A}$$

The household's FONC can be written

$$\frac{U_m(\bar{c},\bar{m},1-\bar{n})}{U_c(\bar{c},\bar{m},1-\bar{n})} = \frac{\gamma/\bar{m}}{1/\bar{c}} = \frac{\bar{\imath}}{1+\bar{\imath}} \tag{1}$$

$$\frac{U_\ell(\bar{c},\bar{m},1-\bar{n})}{U_c(\bar{c},\bar{m},1-\bar{n})} = \frac{\eta/(1-\bar{n})}{1/\bar{c}} = \bar{w} \tag{2}$$

We already know what $\bar{\imath}$ and $\bar{w}$ are. To solve for the triple $(\bar{c},\bar{m},\bar{n})$, we need one more equation. The required condition is *goods market clearing*

$$\bar{y} = \bar{c} = \bar{A}\bar{n} \tag{3}$$

We can now solve for steady state labor $\bar{n}$ by plugging (3) into (2) to get

$$\eta\bar{A}\bar{n} = (1-\bar{n})\bar{w} = (1-\bar{n})\bar{A}$$

The $\bar{A}$ cancel from each side and we are left with

$$\bar{n} = \frac{1}{1+\eta}$$

Hence from (3), we have output and consumption

$$\bar{y} = \bar{c} = \bar{A}\frac{1}{1+\eta}$$

We can now solve for real balances $\bar{m}$ by plugging this into (1) to get

$$\frac{\gamma}{\bar{m}} = \frac{\bar{\imath}}{1+\bar{\imath}}\frac{1}{\bar{c}} = \left(\frac{1+\overline{\mu}-\beta}{1+\overline{\mu}}\right)\frac{1+\eta}{\bar{A}}$$

Hence

$$\bar{m} = \frac{1+\bar{\mu}}{1+\bar{\mu}-\beta}\frac{\gamma\bar{A}}{1+\eta}$$

Finally, we can solve for the initial price level P_0 using

$$\bar{m} = \frac{M_1}{P_0} = \frac{(1+\bar{\mu})M_0}{P_0}$$

so that

$$P_0 = (1+\bar{\mu})\left(\frac{M_0}{\bar{m}}\right) = (1+\eta)(1+\bar{\mu}-\beta)\left(\frac{M_0}{\gamma\bar{A}}\right)$$

(e) (10 marks). The log-linear model can be written

$$\begin{aligned}
\hat{m}_t &= -\eta\hat{\imath}_t + \hat{c}_t \\
E_t\{\hat{c}_{t+1} - \hat{c}_t\} &= \hat{\imath}_t - E_t\{\hat{\pi}_{t+1}\} \\
\hat{m}_{t+1} &= \hat{m}_t + \hat{\mu}_{t+1} - \hat{\pi}_{t+1} \\
\hat{w}_t &= \hat{A}_t \\
\hat{c}_t &= \hat{y}_t = \hat{A}_t + \hat{n}_t \\
\hat{n}_t &= 0
\end{aligned}$$

plus the exogenous shocks. In this system, the constant η comes from the log-linearization. It is given by $\eta = (1+\bar{\imath})/\bar{\imath}^2$ where $\bar{\imath}$ is the steady state nominal interest rate that we've already solved for. The key property of this model is that the labor supply in equilibrium is constant in and out of steady state. To see this, note that with the assumed preferences, the labor supply FONC can always be written

$$\frac{\eta/(1-n_t)}{1/c_t} = w_t = A_t$$

or

$$\eta\frac{c_t}{1-n_t} = w_t = A_t$$

Goods market clearing requires

$$y_t = c_t = A_t n_t$$

Hence

$$\eta\frac{A_t n_t}{1-n_t} = w_t = A_t$$

So n_t is a constant always

$$n_t = \frac{1}{1+\eta}$$

and its log deviation is of course $\hat{n}_t = 0$. This model reduces exactly to that studied in Homework #2 with the technology shocks playing the role of endowment shocks.

In equilibrium, with $\hat{c}_t = \hat{A}_t$ and with $E_t\{\hat{A}_{t+1}\} = \rho_A \hat{A}_t$, these equations can be simplified to

$$\hat{m}_t = -\eta \hat{\imath}_t + \hat{A}_t \tag{4}$$

$$(\rho_A - 1)\hat{A}_t = \hat{\imath}_t - E_t\{\hat{\pi}_{t+1}\} \tag{5}$$

$$\hat{m}_{t+1} = \hat{m}_t + \hat{\mu}_{t+1} - \hat{\pi}_{t+1} \tag{6}$$

These equations describe a *linear rational expectations* model in a single endogenous state variable $\hat{m}_t$. If we can find a solution for $\hat{m}_t$ in terms of the exogenous state variables $\hat{A}_t$ and $\hat{\mu}_t$ we'll be done. To do this, begin by rewriting (6) as

$$\hat{\pi}_{t+1} = -(\hat{m}_{t+1} - \hat{m}_t) + \hat{\mu}_{t+1}$$

Taking conditional expectations on both sides gives

$$E_t\{\hat{\pi}_{t+1}\} = -E_t\{\hat{m}_{t+1} - \hat{m}_t\} + \rho_\mu \hat{\mu}_t$$

This then implies from the bond pricing Euler equation

$$(\rho_A - 1)\hat{A}_t = \hat{\imath}_t + E_t\{\hat{m}_{t+1} - \hat{m}_t\} - \rho_\mu \hat{\mu}_t$$

Use the money demand condition (4) to eliminate the nominal interest rate gives

$$(\rho_A - 1)\hat{A}_t = \frac{1}{\eta}\left(\hat{A}_t - \hat{m}_t\right) + \hat{A}_t + E_t\{\hat{m}_{t+1} - \hat{m}_t\} - \rho_\mu \hat{\mu}_t$$

And simplifying gives

$$\begin{aligned} \hat{m}_t &= \frac{\eta}{1+\eta} E_t\{\hat{m}_{t+1}\} + \frac{\eta}{1+\eta}\hat{x}_t \\ \hat{x}_t &\equiv \frac{\eta(1-\rho_A)+1}{\eta}\hat{A}_t - \rho_\mu \hat{\mu}_t \end{aligned}$$

This is a linear stochastic difference equation in one unknown variable $\hat{m}_t$ with exogenous forcing process $\hat{x}_t$. Once we've solved it, we're largely done. By iterating forwards recursively and imposing the transversality condition, it's easy to show that the unique bounded solution to this stochastic difference equation is

$$\begin{aligned} \hat{m}_t &= E_t\left\{\frac{\eta}{1+\eta}\sum_{s=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^s \hat{x}_{t+s}\right\} \\ &= E_t\left\{\frac{\eta}{1+\eta}\sum_{s=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^s \left[\frac{\eta(1-\rho_A)+1}{\eta}\hat{A}_{t+s} - \rho_\mu \hat{\mu}_{t+s}\right]\right\} \end{aligned}$$

Although this constitutes a solution to the difference equation and allows us to calculate lots of things, it's a little more attractive to simplify this answer by using the linear nature of the shock processes to compute the conditional expectations. Noting that

$$\begin{aligned} E_t\{\hat{A}_{t+s}\} &= \rho_A^s \hat{y}_t \\ E_t\{\hat{\mu}_{t+s}\} &= \rho_\mu^s \hat{\mu}_t \end{aligned}$$

and computing the infinite sums gives

$$\begin{aligned} \hat{m}_t &= E_t\left\{\frac{\eta}{1+\eta}\sum_{s=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^s\left[\frac{\eta(1-\rho_A)+1}{\eta}\hat{A}_{t+s}-\rho_\mu\hat{\mu}_{t+s}\right]\right\} \\ &= \frac{\eta}{1+\eta}\sum_{s=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^s\left[\frac{\eta(1-\rho_A)+1}{\eta}\rho_A^s\hat{A}_t-\rho_\mu^{s+1}\hat{\mu}_t\right] \\ &= \hat{A}_t-\frac{\rho_\mu\eta}{1+\eta-\rho_\mu\eta}\hat{\mu}_t \end{aligned}$$

This gives the endogenous state variable $\hat{m}_t$ as a function of the exogenous state variables $\hat{A}_t$ and $\hat{\mu}_t$. Relatively high money growth reduces real balances through the expected inflation channel. Relatively high output increases money demand. Plugging this result back into the equations above allows us to solve for other endogenous variables, in this case, nominal interest rates and inflation

$$\begin{aligned} \hat{\pi}_{t+1} &= -(\hat{m}_{t+1}-\hat{m}_t)+\hat{\mu}_{t+1} \\ &= -(\hat{A}_{t+1}-\hat{A}_t)+\hat{\mu}_{t+1}-\frac{\rho_\mu\eta}{1+\eta-\rho_\mu\eta}(\hat{\mu}_t-\hat{\mu}_{t+1}) \end{aligned}$$

This implies

$$E_t\{\hat{\pi}_{t+1}\}=(1-\rho_A)\hat{A}_t+\frac{\rho_\mu}{1+\eta-\rho_\mu\eta}\hat{\mu}_t$$

The first of these terms is just the expected growth rate of output, while the second is the expected growth rate of nominal money adjusted for changes in real balances. Notice that this implies

$$\begin{aligned} \hat{\imath}_t &= (\rho_A-1)\hat{A}_t+E_t\{\hat{\pi}_{t+1}\} \\ &= \frac{\rho_\mu}{1+\eta-\rho_\mu\eta}\hat{\mu}_t \end{aligned}$$

so the nominal interest rate just depends on money growth via expected inflation.

Chris Edmond, 15 October 2003