316-466 Monetary Economics — Note 1

Let's turn to models with uncertainty. To begin with, I will revert to the simple two-period model. There are two dates, $t = 0$ and $t = 1$. However, in the second period there are many possible *states* of the world. For simplicity, suppose that there are $S > 1$ possible states with associated probabilities $\pi(s) > 0$ for each $s = 1, ..., S$. Obviously, these are assumed to satisfy

$$\sum_{s=1}^{S} \pi(s) = 1$$

A consumer i is endowed with an amount y_0^i of a single state contingent commodity in the first period. She is also endowed with the vector y_1^i that describes what her endowment of the commodity is if the state s is realized

$$y_1^i = \left(y_1^i(1), \cdots, y_1^i(s), \cdots y_1^i(S)\right)$$

The preferences of each consumer $U^i(c^i)$ over consumption realizations $c^i = (c_0^i, c_1^i)$ are described by an expected-utility function of the form

$$U^i(c^i) = u^i(c_0^i) + \beta E\{u^i(c_1^i)\}$$

In this expression, the notation $E\{u\}$ denotes the mathematical expectation of a random variable u. It is often very useful to spell the expectation out explicitly so that, in this case,

$$\begin{aligned} U^i(c^i) &= u^i(c_0^i) + \beta_i E\{u(c_1^i)\} \\ &= u^i(c_0^i) + \beta_i \sum_{s=1}^{S} u^i[c_1^i(s)]\pi(s) \end{aligned}$$

Notice that every individual i is assumed to assign the same probability $\pi(s)$ to state s.

Let $q_1(s)$ denote the price as of date zero of the single commodity if delivered at date one if state s occurs. Then the budget constraint of consumer i can be written

$$c_0^i + \sum_{s=1}^{S} q_1(s)c_1^i(s) \leq y_0^i + \sum_{s=1}^{S} q_1(s)y_1^i(s)$$

Note that no probabilities enter this constraint. The budget constraint has to hold at every date and state.

The Lagrangian for this problem is

$$L^i = u^i(c_0^i) + \beta_i \sum_{s=1}^{S} u^i[c_1^i(s)]\pi(s) + \lambda^i \left[y_0^i - c_0^i + \sum_{s=1}^{S} q_1(s)(y_1^i(s) - c_1^i(s)) \right]$$

There are $S + 1$ first order conditions. Specifically,

$$(u^i)'(c_0^i) = \lambda^i$$

and for each $s = 1, ..., S$

$$\beta_i (u^i)'[c_1^i(s)]\pi(s) = \lambda^i q_1(s)$$

With the budget constraint, these $S+2$ equations can (implicitly) be solved for the $S+2$ unknowns, namely (c^i, λ^i). Let's examine a couple of important implications of these optimality conditions. First, for any state s

$$\beta_i \frac{(u^i)'[c_1^i(s)]}{(u^i)'(c_0^i)}\pi(s) = q_1(s)$$

(This should look suspiciously similar to the formula we derived for intertemporal prices in the model with no uncertainty). Hence the price system aligns consumers' marginal rates of substitution. This will deliver strong *risk-sharing* implications. Second, for any two states s and z

$$\frac{(u^i)'[c_1^i(s)]\pi(s)}{(u^i)'[c_1^i(z)]\pi(z)} = \frac{q_1(s)}{q_1(z)}$$

This is the familiar equalization of marginal rates of substitution with relative prices.

Finally, denote the price of a riskless bond by p. This is bond that pays one unit of consumption no matter which state occurs and can be synthesized by buying a claim to a unit of consumption for every possible state. Obviously, this has the price

$$p = \sum_{s=1}^{S} q_1(s)$$

Now summing $\beta_i (u^i)'[c_1^i(s)]\pi(s)$ over the states we get

$$\beta_i \sum_{s=1}^{S} (u^i)'[c_1^i(s)]\pi(s) = \lambda^i \sum_{s=1}^{S} q_1(s)$$

or

$$\beta_i E\{(u^i)'(c_1^i)\} = \lambda^i p$$

If the real interest rate r associated with this bond is given by $p = (1+r)^{-1}$, we have the familiar consumption smoothing equation across *dates*.

$$(u^i)'(c_0^i) = \beta_i (1+r) E\{(u^i)'(c_1^i)\}$$

Example 1.

Here is a simple example where everything can be cranked out by hand. Suppose that there are two consumers, $i = 1, 2$, and two states of nature, $s = 1, 2$ with $\pi(1) \equiv \pi$ and $\pi(2) \equiv 1 - \pi$. Suppose further that both consumers have identical Cobb-Douglas preferences

$$\begin{aligned} U(c^i) &= \log(c_0^i) + \beta E\{\log(c_1^i)\} \\ &= \log(c_0^i) + \beta\pi \log[c_1^i(1)] + \beta(1-\pi)\log[c_1^i(2)] \end{aligned}$$

Denote the intertemporal wealth of consumer i facing state prices $q = (q_1(1), q_1(2))$ by $W^i(q)$,

$$W^i(q) = y_0^i + q_1(1)y_1^i(1) + q_1(2)y_1^i(2)$$

Solving the consumer's optimization problem leads to the demand functions

$$\begin{aligned}
\hat{c}_0^i(q) &= \frac{1}{1+\beta}\frac{W^i(q)}{1} \\
\hat{c}_1^i(1;q) &= \frac{\beta}{1+\beta}\pi\frac{W^i(q)}{q_1(1)} \\
\hat{c}_1^i(2;q) &= \frac{\beta}{1+\beta}(1-\pi)\frac{W^i(q)}{q_1(2)}
\end{aligned}$$

To find equilibrium state prices, we need to clear markets in each state. That is, we need to find a price vector $q = (q_1(1), q_1(2))$ such that

$$\begin{aligned}
\hat{c}_1^1(1;q) + \hat{c}_1^2(1;q) &= y_1^1(1) + y_1^2(1) \\
\hat{c}_1^1(2;q) + \hat{c}_1^2(2;q) &= y_1^1(2) + y_1^2(2)
\end{aligned}$$

The attractive feature of log preferences is that this system is a set of linear equations in the unknowns. Substituting for the demand functions and rearranging

$$\begin{aligned}
\frac{\beta}{1+\beta}\pi\left[W^1(q) + W^2(q)\right] &= q_1(1)[y_1^1(1) + y_1^2(1)] \\
\frac{\beta}{1+\beta}(1-\pi)\left[W^1(q) + W^2(q)\right] &= q_1(2)[y_1^1(2) + y_1^2(2)]
\end{aligned}$$

And substituting the definitions of wealth

$$\begin{aligned}
\frac{\beta}{1+\beta}\pi\{y_0^1 + y_0^2 + q_1(1)[y_1^1(1) + y_1^2(1)] + q_1(2)[y_1^1(2) + y_1^2(2)]\} &= q_1(1)[y_1^1(1) + y_1^2(1)] \\
\frac{\beta}{1+\beta}(1-\pi)\{y_0^1 + y_0^2 + q_1(1)[y_1^1(1) + y_1^2(1)] + q_1(2)[y_1^1(2) + y_1^2(2)]\} &= q_1(2)[y_1^1(2) + y_1^2(2)]
\end{aligned}$$

Rearranging gives

$$\begin{aligned}
q_1(1) &= w_1\frac{\sum_i y_1^i(2)}{\sum_i y_1^i(1)}q_1(2) + w_1\frac{\sum_i y_0^i}{\sum_i y_1^i(1)} \\
q_1(2) &= w_2\frac{\sum_i y_1^i(1)}{\sum_i y_1^i(2)}q_1(1) + w_2\frac{\sum_i y_0^i}{\sum_i y_1^i(2)}
\end{aligned}$$

where the weights are

$$w_1 \equiv \frac{\beta\pi}{1+\beta(1-\pi)}, \qquad w_2 \equiv \frac{\beta(1-\pi)}{1+\beta\pi}$$

Solving gives the equilibrium state prices

$$\begin{aligned}
q_1(1) &= \frac{w_1 + w_1w_2}{1 - w_1w_2}\frac{\sum_i y_0^i}{\sum_i y_1^i(1)} \\
q_1(2) &= \frac{w_2 + w_1w_2}{1 - w_1w_2}\frac{\sum_i y_0^i}{\sum_i y_1^i(2)}
\end{aligned}$$

The equilibrium prices are decreasing in the aggregate supply of the commodity in their respective states. The prices are increasing in both w_1 and w_2. In turn, w_1 and w_2 are both increasing in β while w_1 is increasing in π but w_2 is decreasing in π. Hence, 1) the more patient are the consumers, the more valuable is second period consumption and so the higher are both state prices; and 2) the more likely is state $s = 1$ (the higher is π) the more valuable is a claim to consumption in state 2 and so the higher is the price for state 2.

(Of course these prices are positive, $0 < w_1 w_2 < 1$ because

$$w_1 w_2 = \frac{\beta\pi}{1+\beta\pi}\frac{\beta(1-\pi)}{1+\beta(1-\pi)} < 1$$

follows from $0 < \beta < 1$ and $0 < \pi < 1$).

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