316-466 Monetary Economics — Note 3

In this note, I discuss some handy facts about steady states and local approximations to them in the context of the Solow (1956) growth model.

Setup

Consider the Solow growth model in its most stripped down form. This consists of a resource constraint, a production function, a law of motion for capital accumulation, and the behavioral assumption that a constant fraction $0 < s < 1$ of output is saved and invested. In standard notation

$$\begin{aligned} c_t + i_t &= y_t = f(k_t) \\ k_{t+1} &= (1-\delta)k_t + i_t \\ c_t &= (1-s)y_t \end{aligned}$$

for some constant depreciation rate $0 < \delta < 1$ and given initial condition k_0.

The following assumptions are made about the production function: It is strictly increasing, strictly concave, needs some capital to produce anything, and satisfies the Inada conditions

$$\begin{aligned} f(k) &\geq 0, \qquad f(0) = 0 \\ f'(k) &> 0, \qquad f''(k) < 0 \\ f'(0) &= \infty, \qquad f'(\infty) = 0 \end{aligned}$$

Scalar difference equation

Putting this together gives a first order non-linear difference equation

$$k_{t+1} = H(k_t) \equiv sf(k_t) + (1-\delta)k_t$$

Given the assumptions about f, the function H is also strictly increasing, strictly concave, etc

$$\begin{aligned} H(0) &= sf(0) + (1-\delta)0 = 0 \\ H'(k) &= sf'(k) + (1-\delta) > 0 \\ H''(k) &= sf''(k) < 0 \\ H'(0) &= sf'(0) + (1-\delta) = \infty \\ H'(\infty) &= sf'(\infty) + (1-\delta) = (1-\delta) < 1 \end{aligned}$$

This model has a single *state* variable, the capital stock k_t. Given the capital stock, it's trivial to find output $y_t = f(k_t)$, consumption $c_t = (1-s)f(k_t)$, and investment $i_t = sf(k_t)$.

A *solution* to a difference equation is a function that expresses the capital stock as a function of time, t an initial condition, k_0 and other parameters of the model. *Linear* difference equations are trivial to solve. For example, suppose

$$x_{t+1} = Ax_t$$

Then the solution is found by recursive substitution

$$\begin{aligned} x_1 &= Ax_0 \\ x_2 &= Ax_1 = AAx_0 = A^2x_0 \\ &\vdots \\ x_t &= A^tx_0 \end{aligned}$$

Unfortunately, non-linear difference equations are not so easy to solve. It's traditional to analyze their qualitative behavior in terms of steady states and phase diagrams.

Steady states

A steady state is a *fixed point* of the mapping $k_{t+1} = H(k_t)$. Put differently, it is a situation where the state variable is unchanging so that $k_{t+1} = k_t = k$. The Solow model has two fixed points. A steady state has to satisfy

$$k = H(k) = sf(k) + (1-\delta)k$$

One solution is $k = 0$

$$0 = H(0) = sf(0) + (1-\delta)0$$

another is the unique positive solution to

$$\delta k = sf(k)$$

The left hand side is a straight line through the origin with positive slope $0 < \delta < 1$. This is the steady state level of investment, the amount of investment required to keep the capital stock constant. The right hand side is a strictly increasing strictly concave function that is exactly a scalar multiple of the production function $f(k)$. Given the Inada conditions, there is a unique positive $\bar{k} > 0$ determined by the intersection of the two curves $\delta\bar{k} = sf(\bar{k})$.

Local stability

Consider the first order Taylor series approximation to the function H at some point $\tilde{k}$

$$H(k_t) \cong H(\tilde{k}) + H'(\tilde{k})(k_t - \tilde{k})$$

A natural point around which to approximate H is a fixed point. In this case,

$$\begin{aligned} H(k_t) &\cong H(\bar{k}) + H'(\bar{k})(k_t - \bar{k}) \\ &= \bar{k} + H'(\bar{k})(k_t - \bar{k}) \end{aligned}$$

where the second line follows because of the definition of $\bar{k}$ as a point that satisfies $\bar{k} = H(\bar{k})$. If we treat this approximation as exact, we have the linear difference equation

$$k_{t+1} = \bar{k} + H'(\bar{k})(k_t - \bar{k})$$

An intuitive understanding of whether a fixed point is locally stable can be obtained by considering what would have to be true in order for sequences of k_t that satisfy this linear difference equation to converge to $\bar{k}$. Writing $x_{t+1} = k_{t+1} - \bar{k}$ so that $x_{t+1} = H'(\bar{k})x_t$, it is clear that this linear difference equation has the solution

$$x_t = H'(\bar{k})^t x_0$$

or

$$k_t = \bar{k} + H'(\bar{k})^t (k_0 - \bar{k})$$

Clearly, $k_t \longrightarrow \bar{k}$ if and only if $H'(\bar{k})^t \longrightarrow 0$ and this in turn requires that $\left|H'(\bar{k})\right| < 1$. Thus in order for a fixed point to be locally stable, we need the absolute value of the slope of H at the fixed point to be less than one. If $0 < H'(\bar{k}) < 1$, the convergence to the fixed point is *monotone*. If $-1 < H'(\bar{k}) < 0$, the convergence to the fixed point takes the form of *dampened oscillations*.

Consider the Solow model. The trivial fixed point $\bar{k} = 0$ is locally unstable — it *repels* capital sequences — because the derivative at this point is $H'(0) = \infty$. On the other hand, the interior fixed point is locally stable — it *attracts* capital sequences — because the derivative at this point is positive but less than one.

Of course, just because a fixed point is locally stable, doesn't mean that it is globally stable (that would require $k_t \longrightarrow \bar{k}$ for all initial conditions k_0 in the domain of the original non-linear map H).

Log-linearizations

Log-linearizing a model has the convenience of providing coefficients that are readily interpretable as *elasticities*. The log deviation of a variable x_t from a steady state level $\bar{x}$ is just $\hat{x}_t \equiv \log(x_t/\bar{x})$. Multiplied by 100, this is approximately the percentage deviation of x_t from $\bar{x}$. Mechanically, the log linearization of a model proceeds by replacing x_t with $\bar{x}\exp(\hat{x}_t)$ and then linearizing the equations of the model with respect to $\hat{x}_t$ in a neighborhood of zero.

To illustrate, consider the fundamental non-linear difference equation of the Solow model

$$k_{t+1} = H(k_t) \equiv sf(k_t) + (1-\delta)k_t$$

Replace k_t by $\bar{k}\exp(\hat{k}_t)$ to get

$$\bar{k}\exp(\hat{k}_{t+1}) = H[\bar{k}\exp(\hat{k}_t)] \equiv sf[\bar{k}\exp(\hat{k}_t)] + (1-\delta)\bar{k}\exp(\hat{k}_t)$$

A first order Taylor series approximation around zero of the term on the far left gives

$$\bar{k}\exp(\hat{k}_{t+1}) \cong \bar{k}\exp(0) + \bar{k}\exp(0)\hat{k}_{t+1} = \bar{k} + \bar{k}\hat{k}_{t+1} \tag{1}$$

Similarly

$$\begin{aligned} H[\bar{k}\exp(\hat{k}_t)] &\cong H[\bar{k}\exp(0)] + H'[\bar{k}\exp(0)]\bar{k}\exp(0)\hat{k}_t \\ &= H(\bar{k}) + H'(\bar{k})\bar{k}\hat{k}_t \\ &= \bar{k} + [sf'(\bar{k}) + (1-\delta)]\bar{k}\hat{k}_t \end{aligned} \tag{2}$$

Equating (1) and (2) and simplifying gives

$$\hat{k}_{t+1} \cong [sf'(\bar{k}) + (1-\delta)]\hat{k}_t$$

Here are some rules that make taking log-linearization easier (you should derive these results yourself to test your understanding).

1. MULTIPLICATION

$$\begin{aligned} z_t &= x_t y_t \\ &\Longrightarrow \hat{z}_t = \hat{x}_t + \hat{y}_t \end{aligned}$$

Application: A consumer's first order condition is often

$$1 = \beta \frac{U'(c_{t+1})}{U'(c_t)} R_{t+1}$$

This implies the log linearization

$$\hat{1} = \hat{\beta} + \widehat{U'(c_{t+1})} - \widehat{U'(c_t)} + \hat{R}_{t+1}$$

But since the log deviations of constants are zero, this is just

$$0 = \widehat{U'(c_{t+1})} - \widehat{U'(c_t)} + \hat{R}_{t+1}$$

2. DIVISION

$$\begin{aligned} z_t &= x_t / y_t \\ &\Longrightarrow \hat{z}_t = \hat{x}_t - \hat{y}_t \end{aligned}$$

Application: The law of motion for money supply is often written

$$M_{t+1} = \mu_t M_t$$

This implies the law of motion for *real balances*

$$m_{t+1} \equiv \frac{M_{t+1}}{P_t} = \mu_t \frac{M_t}{P_{t-1}} \frac{1}{\pi_t}$$

(where $\pi_t = P_t / P_{t-1}$). So we have the log linearization

$$\hat{m}_{t+1} = \hat{\mu}_t + \hat{m}_t - \hat{\pi}_t$$

3. ADDITION/SUBTRACTION

$$\begin{aligned} z_t &= x_t + y_t \\ &\Longrightarrow \bar{z}\hat{z}_t = \bar{x}\hat{x}_t + \bar{y}\hat{y}_t \end{aligned}$$

Applications: The resource constraint is often

$$c_t + i_t = y_t$$

This implies the log linearization

$$\bar{c}\hat{c}_t + \bar{\imath}\hat{\imath}_t = \bar{y}\hat{y}_t$$

Another example is that gross returns are often written as one plus net returns

$$R_{t+1} = 1 + r_{t+1}$$

This implies

$$\bar{R}\hat{R}_{t+1} = \bar{r}\hat{r}_{t+1}$$

or

$$\hat{R}_{t+1} = \frac{\bar{r}}{1+\bar{r}}\hat{r}_{t+1}$$

4. SMOOTH FUNCTIONS

$$\begin{aligned} z_t &= f(x_t) \\ &\Longrightarrow \bar{z}\hat{z}_t = f'(\bar{x})\bar{x}\hat{x}_t \end{aligned}$$

Applications: The marginal utility of consumption may be $U'(c)$.This implies the log linearization

$$U'(\bar{c})\widehat{U'(c_t)} = U''(\bar{c})\bar{c}\hat{c}_t$$

As a leading example, if we have constant relative risk aversion preferences with coefficient $\sigma > 0$, this implies

$$\widehat{U'(c_t)} = \frac{U''(\bar{c})\bar{c}}{U'(\bar{c})}\hat{c}_t = -\sigma\hat{c}_t$$

As a further example, if we have the rental rate of capital equal to its marginal product

$$r_t = F'(k_t)$$

this implies

$$\hat{r}_t = \frac{F''(\bar{k})\bar{k}}{F'(\bar{k})}\hat{k}_t$$

5. MULTIVARIATE SMOOTH FUNCTIONS

$$\begin{aligned} z_t &= f(x_t, y_t) \\ &\Longrightarrow \bar{z}\hat{z}_t = f_x(\bar{x},\bar{y})\bar{x}\hat{x}_t + f_y(\bar{x},\bar{y})\bar{y}\hat{y}_t \end{aligned}$$

Applications: The production function might be $y = F(k,n)$. This implies the log linearization

$$\bar{y}\hat{y}_t = F_k(\bar{k},\bar{n})\bar{k}\hat{k}_t + F_n(\bar{k},\bar{n})\bar{n}\hat{n}_t$$

or

$$\hat{y}_t = \frac{F_k(\bar{k}, \bar{n})\bar{k}}{F(\bar{k}, \bar{n})}\hat{k}_t + \frac{F_n(\bar{k}, \bar{n})\bar{n}}{F(\bar{k}, \bar{n})}\hat{n}_t$$

As a leading example, if we have a Cobb-Douglas production function $F(k, n) = k^\alpha n^{1-\alpha}$ for $0 < \alpha < 1$, this is

$$\hat{y}_t = \alpha\hat{k}_t + (1 - \alpha)\hat{n}_t$$

As a further example, if we have the marginal utility of consumption $U_c(c, \ell)$, the associated log linearization is

$$U_c\widehat{(c_t, \ell_t)} = \frac{U_{cc}(\bar{c}, \bar{\ell})\bar{c}}{U_c(\bar{c}, \bar{\ell})}\hat{c}_t + \frac{U_{c\ell}(\bar{c}, \bar{\ell})\bar{\ell}}{U_c(\bar{c}, \bar{\ell})}\hat{\ell}_t$$

And if we use the approximation implied by Rule 3 to write $\bar{n}\hat{n}_t + \bar{\ell}\hat{\ell}_t = 0$ with $\bar{\ell} = 1 - \bar{n}$, we can also write

$$U_c\widehat{(c_t, \ell_t)} = \frac{U_{cc}(\bar{c}, 1 - \bar{n})\bar{c}}{U_c(\bar{c}, 1 - \bar{n})}\hat{c}_t - \frac{U_{c\ell}(\bar{c}, 1 - \bar{n})\bar{n}}{U_c(\bar{c}, 1 - \bar{n})}\hat{n}_t$$

Chris Edmond, 9 September 2003